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ON A CONJECTURE CONCERNING THE HARMONIC SERIES

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Let

$$H_k = \sum_{i=1}^{k} \frac{1}{i}.$$

The Euler constant is defined as (see, e.g., [1]):

$$C = \lim_{k\to\infty} \sum_{i=1}^{k} (\frac{1}{i} - \ln \frac{i+1}{i}) = 0.5772156649....$$

Hence,

$$C = H_k - \ln(1+k) + \alpha_k,$$

where $\alpha_k > 0$ and then

$$H_k = \ln(1+k) + C + \gamma_k, \quad (1)$$

where $\gamma_k = -\alpha_k < 0$.

For each integer $k > 1$ there exists a corresponding integer n_k such that

$$H_{n_k - 1} < k < H_{n_k}.$$

In [2] it is posed as a problem to prove that

$$n_k = [e^{k-C}] \quad \text{or} \quad n_k = [e^{k-C}] + 1, \quad (2)$$

where $[x]$ denotes the integer part of the real number x. A weaker assertion is given by the author together with L. Asenova (now – his wife L. Atanassova) in [3]: for every integer $k \geq 3$:

$$[H_{[\sum_{i=0}^{k} e^i]}] = k + 1.$$

In [1] it is proved that

$$e^{k-C} - \frac{1}{24e^{k-C}} - \frac{2}{e^{3(k-C)}} - \frac{1}{2} < n_k < e^{k-C} - \frac{1}{24e^{k-C}} + \frac{2}{e^{3(k-C)}} + \frac{1}{2} \quad (3)$$

and in [4]: if $e^{k-C} = m + \alpha$, where $m = [e^{k-C}]$ and $0 < \alpha < 1$, then

$$\alpha \notin [\frac{1}{2} - \frac{1}{10m}, \frac{1}{2} + \frac{1}{m}]. \quad (4)$$

Here we shall prove the following

THEOREM 1: For every natural number $k \geq 3$:

$$n_k = [e^{k-C} + \frac{1}{2}]. \tag{5}$$

Proof: Initially, we shall note that

$$\gamma_{k+1} - \gamma_k = H_{k+1} - \ln(k+2) - H_k + \ln(k+1) = \frac{1}{k+1} - \ln(1 + \frac{1}{k+1}) > 0,$$

i.e.

$$\gamma_{k+1} > \gamma_k$$

and the sequence $\{\gamma\}_{k=1}^{\infty}$ is growing.

It is valid the following

LEMMA: For every natural number k:

$$\ln \frac{k + \frac{1}{2} + \frac{1}{2k}}{k+1} > \gamma_k > \ln \frac{k+\frac{1}{2}}{k+1}. \tag{6}$$

We shall use the formula

$$H_k = \ln k + C + \frac{1}{2k} - \frac{1}{12k^2} + \frac{\varepsilon_k}{120k^4}$$

for $0 < \varepsilon_k < 1$ (see formula (6.66) from [5]). From here and from (1) we have that

$$\ln(1+k) + C + \gamma_k = \ln k + C + \frac{1}{2k} - \frac{1}{12k^2} + \frac{\varepsilon_k}{120k^4}$$

i.e.,

$$\gamma_k = -\ln\frac{k+1}{k} + \frac{1}{2k} - \frac{1}{12k^2} + \frac{\varepsilon_k}{120k^4}.$$

Therefore,

$$\ln \frac{k + \frac{1}{2} + \frac{1}{2k}}{k+1} - \gamma_k$$

$$= \ln \frac{k + \frac{1}{2} + \frac{1}{2k}}{k+1} + \ln\frac{k+1}{k} - \frac{1}{2k} + \frac{1}{12k^2} - \frac{\varepsilon_k}{120k^4}$$

$$= \ln \frac{k + \frac{1}{2} + \frac{1}{2k}}{k} - \frac{1}{2k} + \frac{1}{12k^2} - \frac{\varepsilon_k}{120k^4}$$

$$= \ln(1 + \frac{1}{2k} + \frac{1}{2k^2}) - \frac{1}{2k} + \frac{1}{12k^2} - \frac{\varepsilon_k}{120k^4}$$

(from formula (601) from [6])

$$> \frac{1}{2k} + \frac{1}{2k^2} - \frac{1}{2}(\frac{1}{2k} + \frac{1}{2k^2})^2 - \frac{1}{2k} + \frac{1}{12k^2} - \frac{1}{120k^4}$$

$$= \frac{1}{2k^2} - \frac{1}{8k^2} - \frac{1}{4k^3} - \frac{1}{8k^4} + \frac{1}{12k^2} - \frac{1}{120k^4}$$

$$= \frac{11}{24k^2} - \frac{1}{4k^3} - \frac{16}{120k^4}$$

$$= \frac{1}{2k^2}(\frac{11}{12} - \frac{1}{2k} - \frac{4}{15k^2})$$

$$> \frac{1}{2k^2}(\frac{11}{12} - \frac{1}{2} - \frac{4}{15}) > 0$$

and

$$\gamma_k - \ln\frac{2k+1}{2k+2} = -\ln\frac{2k+1}{2k+2} - \ln\frac{k+1}{k} + \frac{1}{2k} - \frac{1}{12k^2} + \frac{\varepsilon_k}{120k^4}$$

$$> -\ln(1+\frac{1}{2k}) + \frac{1}{2k} - \frac{1}{12k^2}$$

(from formula (601.1) from [6])

$$= \frac{1}{2k} - \frac{1}{12k^2} - \frac{1}{2k} + \frac{1}{8k^2} - \frac{1}{24k^3} + \dots$$

$$> \frac{1}{24k^2} - \frac{1}{24k^3} + \dots > 0.$$

with which (6) and therefore the Lemma are proved.

In the proof of the Theorem we shall use the method of induction and only the right inequality from the Lemma.

First, we shall prove that for every integer $k \geq 3$:

$$[H_{[e^{k-C}+\frac{1}{2}]}] = k. \tag{7}$$

When $k = 3$ the assertion is valid:

$$[H_{[e^{3-C}+\frac{1}{2}]}] = [H_{11}] = [3\frac{551}{27720}] = 3.$$

In this case $n_3 = [e^{3-C} + \frac{1}{2}] = 11$.

We assume that (7) is valid for some integer $k \geq 3$, and let

$$\begin{cases} H_{[e^{k-C}+\frac{1}{2}]} - [H_{[e^{k-C}+\frac{1}{2}]}] = \delta \\ \gamma_{[e^{k+1-C}+\frac{1}{2}]} - \gamma_{[e^{k-C}+\frac{1}{2}]} = \eta. \end{cases} \tag{8}$$

Obviously, $\delta, \eta > 0$.

Let us denote $a = e^{k+1-C}$. From $k > 3$ follows that $a > 11$). It is necessary to show that:

$$0 < \delta + \eta + \ln\frac{[a+\frac{1}{2}]+1}{e([a+\frac{1}{2}]+1)} < 1. \tag{9}$$

But

$$\delta+\eta+\ln\frac{[a+\frac{1}{2}]+1}{\epsilon([a+\frac{1}{2}]+1)}$$

$$=\gamma_{[a+\frac{1}{2}]}-\gamma_{[\frac{a}{e}+\frac{1}{2}]}+\ln([a+\frac{1}{2}]+1)-\ln e([\frac{a}{e}+\frac{1}{2}]+1)+\gamma_{[\frac{a}{e}+\frac{1}{2}]}+C+\ln([\frac{a}{e}+\frac{1}{2}]+1)-k$$

$$=\gamma_{[a+\frac{1}{2}]}+\ln([a+\frac{1}{2}]+1)-\ln([\frac{a}{e}+\frac{1}{2}]+1)-1+C+\ln([\frac{a}{e}+\frac{1}{2}]+1)-k$$

$$=\gamma_{[a+\frac{1}{2}]}+\ln([a+\frac{1}{2}]+1)-1+C-k$$

$$=\gamma_{[a+\frac{1}{2}]}+\ln([a+\frac{1}{2}]+1)-\ln a$$

$$=\gamma_{[a+\frac{1}{2}]}+\ln\frac{([a+\frac{1}{2}]+1)}{a}.$$

Let $a=m+\alpha$, where $0<\alpha<1$. If $\alpha<\frac{1}{2}$, then:

$$\gamma_{[a+\frac{1}{2}]}+\ln\frac{[a+\frac{1}{2}]+1}{a}=\gamma_m+\ln\frac{m+1}{a}$$

(from the Lemma)

$$>\ln\frac{m+\frac{1}{2}}{m+1}+\ln\frac{m+1}{m+\frac{1}{2}}=0.$$

If $\alpha\geq\frac{1}{2}$, then

$$\gamma_{[a+\frac{1}{2}]}+\ln\frac{[a+\frac{1}{2}]+1}{a}=\gamma_{m+1}+\ln\frac{m+2}{a}$$

(from the Lemma)

$$>\ln\frac{m+1+\frac{1}{2}}{m+2}+\ln\frac{m+2}{m+1}=\ln\frac{m+\frac{3}{2}}{m+1}>0.$$

On the other hand,

$$\gamma_{[a+\frac{1}{2}]}+\ln\frac{[a+\frac{1}{2}]+1}{a}<\ln\frac{a+\frac{3}{2}}{a}=\ln(1+\frac{3}{2a})<\frac{3}{2a}<\frac{3}{22}<1.$$

Therefore, (9) is valid.

Hence, from (8), (1), (7) and (9), respective, it follows that

$$[H_{[e^{k+1-C}+\frac{1}{2}]}]=[H_{[e^{k-C}+\frac{1}{2}]}+H_{[e^{k+1-C}+\frac{1}{2}]}-H_{[e^{k-C}+\frac{1}{2}]}]$$

$$=[k+\delta+\gamma_{[e^{k+1-C}+\frac{1}{2}]}+\ln([e^{k+1-C}+\frac{1}{2}]+1)+C-\gamma_{[e^{k-C}+\frac{1}{2}]}-\ln([e^{k-C}+\frac{1}{2}]+1)-C]$$

$$= k + 1 + [\delta + \eta + \ln \frac{[e^{k+1-C} + \frac{1}{2}] + 1}{e([e^{k-C} + \frac{1}{2}] + 1)}]$$

$$= k + 1.$$

Obviously, $n_k = [e^{k-C} + \frac{1}{2}]$ satisfies the inequalities (cf. (2)):

$$[e^{k-C}] \leq [e^{k-C} + \frac{1}{2}] \leq [e^{k-C}] + 1.$$

We shall prove that n_k satisfies inequalities (3). Let $e^{k-C} = m + \alpha$, where $m = [e^{k-C}]$ and $0 < \alpha < 1$. Then, if $\alpha < \frac{1}{2}$:

$$m + \alpha - \frac{1}{24(m+\alpha)} - \frac{2}{(m+\alpha)^3} - \frac{1}{2} < m = [m + \alpha + \frac{1}{2}] = n_k;$$

if $1 > \alpha \geq \frac{1}{2}$:

$$m + \alpha - \frac{1}{24(m+\alpha)} - \frac{2}{(m+\alpha)^3} - \frac{1}{2} < m + 1 = [m + \alpha + \frac{1}{2}] = n_k.$$

On the other hand, if $\alpha < \frac{1}{2}$:

$$m + \alpha - \frac{1}{24(m+\alpha)} + \frac{2}{(m+\alpha)^3} + \frac{1}{2} > m = [m + \alpha + \frac{1}{2}] = n_k;$$

if $\alpha \geq \frac{1}{2}$, then from (4) it follows that $\alpha > \frac{1}{2} + \frac{1}{m}$ and:

$$m + \alpha - \frac{1}{24(m+\alpha)} + \frac{2}{(m+\alpha)^3} + \frac{1}{2} > m + 1 + \frac{1}{m} - \frac{1}{24m} > m + 1 = [m + \alpha + \frac{1}{2}] = n_k.$$

Therefore, n_k satisfies inequalities (3).

Having in mind that

$$(e^{k-C} - \frac{1}{24e^{k-C}} + \frac{2}{e^{3(k-C)}} + \frac{1}{2}) - (e^{k-C} - \frac{1}{24e^{k-C}} - \frac{2}{e^{3(k-C)}} - \frac{1}{2})$$

$$= \frac{4}{e^{3(k-C)}} + 1 > 1,$$

we must show that only one natural number (i.e., only n_k) satisfies (3). For this aim we check the expressions

$$A \equiv e^{k-C} - \frac{1}{24e^{k-C}} + \frac{2}{e^{3(k-C)}} + \frac{1}{2} - n_k$$

$$B \equiv n_k - (e^{k-C} - \frac{1}{24e^{k-C}} - \frac{2}{e^{3(k-C)}} - \frac{1}{2}).$$

Let again $e^{k-C} = m + \alpha$ for $0 \leq \alpha < 1$ and $m \geq 11$, as we mentioned above.

From (4), if $0 \leq \alpha < \frac{1}{2} - \frac{1}{10m}$, then

$$[e^{k-C} + \frac{1}{2}] = [m + \alpha + \frac{1}{2}]$$

and

$$m = [m + \frac{1}{2}] \leq [m + \alpha + \frac{1}{2}] \leq [m + 1 - \frac{1}{10m}] = m.$$

Therefore,

$$\begin{aligned} A &= m + \alpha - \frac{1}{24(m+\alpha)} + \frac{2}{(m+\alpha)^3} + \frac{1}{2} - m \\ &= \alpha + \frac{1}{2} - \frac{1}{24(m+\alpha)} + \frac{2}{(m+\alpha)^3} \\ &< 1 - \frac{1}{10m} + \frac{2}{(m+\alpha)^3} < 1, \end{aligned}$$

because for $m \geq 11$: $\frac{1}{10} > \frac{2}{m^2}$, and

$$\begin{aligned} B &= m - (m + \alpha - \frac{1}{24(m+\alpha)} - \frac{2}{(m+\alpha)^3} - \frac{1}{2}) \\ &< -\frac{1}{2} + \frac{1}{10m} + \frac{1}{24(m+\alpha)} + \frac{2}{(m+\alpha)^3} + \frac{1}{2} \\ &< \frac{1}{10m} + \frac{1}{24m} + \frac{2}{m^3} < \frac{17}{120m} + \frac{2}{m^3} < 1. \end{aligned}$$

If $\frac{1}{2} + \frac{1}{m} \leq \alpha < 1$, then

$$m + 1 = [m + 1 + \frac{1}{m}] \leq [m + \alpha + \frac{1}{2}] \leq [m + 1 + \frac{1}{2}] = m + 1.$$

Therefore,

$$\begin{aligned} A &= m + \alpha - \frac{1}{24(m+\alpha)} + \frac{2}{(m+\alpha)^3} + \frac{1}{2} - m - 1 \\ &< \frac{1}{2} - \frac{1}{24(m+1)} + \frac{2}{m^3} < 1 \end{aligned}$$

and

$$\begin{aligned} B &= m + 1 - (m + \alpha - \frac{1}{24(m+\alpha)} - \frac{2}{(m+\alpha)^3} - \frac{1}{2}) \\ &< \frac{1}{2} + \frac{1}{24m} + \frac{2}{m^3} < \frac{1}{2} + \frac{1}{240} + \frac{2}{1000} < 1. \end{aligned}$$

Therefore, n_k is the unique natural number in the above mentioned interval and with this the Theorem is proved.

The Theorem is generalized by Mladen Vassilev - Missana in [7] to the form

THEOREM 2: For every $k \in \mathcal{N}$ and $p \in \mathcal{R}^+$ the identity

$$[1 + \frac{1}{1.p+1} + \ldots + \frac{1}{[\frac{e^{kp}-1}{p}].p+1}] = k,$$

i.e.,

$$[H_{n_k(p)}(p)] = k,$$

holds, where $\mathcal{R}^+$ is the set of all positive real numbers,

$$n_k(p) = [\frac{e^{kp}-1}{p}] + 1$$

and

$$H_n(p) = \sum_{i=0}^{n-1} \frac{1}{i.p+1}.$$

The present text is prepared on the basis of [8] and [9] in which there are some mistakes, which kindly were shown to the author by M. Vassilev - Missana and Prof. Donald Knuth.

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