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NOTES ON THE q-STIRLING NUMBERS OF SECOND KIND

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1. INTRODUCTION

Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field and the completion of the algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$.

If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|q-1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. We define the q-analogue of a positive integer n to be $[n] = [n:q] = \frac{q^n - 1}{q-1}$. The q-binomial coefficient for nonnegative integers m and n with $m \geq n$ is

$$\binom{m}{n}_q = \frac{[m]!}{[n]![m-n]!} = \frac{(q^m-1)(q^{m-1}-1)\cdots(q^{m-n+1}-1)}{(q^n-1)(q^{n-1}-1)\cdots(q-1)},$$

where the q-factorials are $[n]! = [n]\cdot[n-1]\cdots[2][1]$, $[0]! = 1$ (see [3]). Define the nth q-power of a polynomial $f(T)$ to be $f^{(0:q)} = 1$ and $f^{(n:q)}(T) = f(T)f(qT)\cdots f(q^{n-1}T)$

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for $n \geq 1$. Then the q-binomial theorem becomes

$$(1) \qquad (1+T)^{(m:q)} = \sum_{k=0}^{m} \binom{m}{k}_q T^{(k:q)}.$$

Let $(Eh)(x) = h(x+1)$ be the shift operator.

Then the q-difference operator is defined by

$$\Delta_q^n := (E-I)^{(n:q)} = \Delta^{(n:q)}.$$

The purpose of this note is to find the generating function of the q-Stirling numbers of second kind using the above q-difference operator. By using this generating function, we can give some formulae on q-Stirling numbers

2. q-ANALOGUE OF STIRLING NUMBERS OF SECOND KIND

For $q \in \mathbb{C}$ with $|q| < 1$, the q-Stirling numbers of second kind were defined by L. Carlitz as a numbers $S_2(n, k : q)$ such that

$$(2) \qquad [x]^n = \sum_{k=0}^{n} q^{\binom{k}{2}} \binom{x}{k}_q [k]! S_2(n, k : q),$$

from which Carlitz found

$$(3) \qquad S_2(n, k : q) = \frac{q^{-\binom{k}{2}}}{[k]!} \sum_{j=0}^{k} (-1)^j q^{\binom{j}{2}} \binom{k}{j}_q [k-j]^n, \quad \text{cf. [1], [3].}$$

Now, we define the operator $*$ on e^t by

$$(4) \qquad f(q) * e^{xt} = f(q) e^{[x]t}.$$

Hence, we have the following:

$$\begin{aligned}
\frac{q^{-\binom{k}{2}}}{[k]!} * (e^t - 1)^{(k:q)} &= \frac{q^{-\binom{k}{2}}}{[k]!} * \sum_{0 \leq j \leq k} (-1)^{k-j} \binom{k}{j}_q q^{\binom{k-j}{2}} e^{jt} \\
&= \frac{q^{-\binom{k}{2}}}{[k]!} \sum_{0 \leq j \leq k} (-1)^{k-j} \binom{k}{j}_q q^{\binom{k-j}{2}} \sum_{n=0}^{\infty} [j]^n \frac{t^n}{n!} \\
&= \sum_{n \geq 0} S_2(n, k : q) \frac{t^n}{n!}, \quad |t| < 1.
\end{aligned}$$

Let $S_2(n,k:q)=0$ if $k>n$. Then we obtain the generating function of the q-Stirling numbers of second kind as follows:

$$\frac{q^{-\binom{k}{2}}}{[k]!} * (e^t-1)^{(k:q)} = \sum_{n\geq k} S_2(n,k:q)\frac{t^n}{n!}, \ |t|<1. \tag{5}$$

For $q=1$, note that

$$(e^t-1)^k = k!\sum_{n\geq k} S_2(n,k)\frac{t^n}{k!}, \ |t|<1 \ ,$$

where $S_2(n,k)$ is the second kind Stirling number .

Now, we assume $q\in\mathbb{C}_p$ with $|1-q|_p<p^{-\frac{1}{p-1}}$.

The q-analogue of Mahler expansion was defined by

$$f(x)=\sum_{n\geq 0}(\Delta_q^n f)(0)\binom{x}{n}_q \in C(\mathbb{Z}_p,\mathbb{C}_p), \text{ cf. [2]}$$

where $C(\mathbb{Z}_p,\mathbb{C}_p)$ denotes the set of continuous functions from $\mathbb{Z}_p$ to $\mathbb{C}_p$.

Moreover

$$(\Delta_q^n f)(0)=\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k q^{\binom{k}{2}} f(n-k), \quad \text{cf. [3].} \tag{6}$$

By (3),(6), note that

$$\Delta_q^k 0^n = \frac{q^{-\binom{k}{2}}}{[k]!} S_2(n,k:q).$$

Let

$$f(x)=\sum_{j=0}^{\infty} a_{j,q}[x]^j, \quad a_{j,q}\in\mathbb{Q}_p. \tag{7}$$

By using (6), (7), it is easy to see that

$$\begin{aligned}(\Delta_q^l f)(0) &= \sum_{m=0}^{l}(-1)^{l-m}\binom{l}{m}_q f(m) q^{\binom{l-m}{2}} \\ &= \sum_{j=0}^{\infty} a_{j,q}\left(\sum_{m=0}^{l}(-1)^{l-m}\binom{l}{m}_q [m]^j q^{\binom{l-m}{2}}\right) \\ &= \sum_{j=0}^{\infty} a_{j,q} q^{\binom{l}{2}}[l]! S_2(j,l:q).\end{aligned}$$

Hence, we have

$$q^{-\binom{l}{2}}\frac{\Delta_q^l f(0)}{[l]!}=\sum_{j=0}^{\infty}a_{j,q}S_2(j,l:q). \tag{8}$$

Remark. By using (8), we easily see that

f is analytic on $\mathbb{Z}_p$ if and only if $|\frac{\Delta_q^n f(0)}{[n]!}|_p \to 0$, as $n\to\infty$.

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