NNTDM 7 (2001), 3, 70-77

TRANSLATION-INVARIANT p-ADIC INTEGRAL ON $\mathbb{Z}_p$

HAE-SOOK CHO AND EUN-SUP KIM

ABSTRACT. In this paper, we treat the some formulas to be related an invariant p-adic integral on $\mathbb{Z}_p$. As an application of an invariant p-adic integral on $\mathbb{Z}_p$, we give the formulas for sums of products of the analogue of Bernoulli numbers to be defined by an invariant p-adic integral on $\mathbb{Z}_p$.

1. INTRODUCTION

Throughout this paper $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and Ω_p will be denoted by the ring of rational integers, the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively.
Let v_p be the normalized exponential valuation of Ω_p with $|p|_p = p^{-v_p(p)} = p^{-1}$.
When one talks of q-extensions, q can be variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \Omega_p$.
We use the notation

$$[x] = [x : q] = \frac{1 - q^x}{1 - q}.$$

2000 *Mathematics Subjct Classfication.*Primary 03B52, 03E72, 94D05
keywords and phrases.

Hence,

$$\lim_{q\to 1}[x:q]=x$$

for any x with $|x|_p \leq 1$ in the present p-adic case. Let d be a fixed integer and let p be a fixed prime number. We set
$X=\varprojlim(\mathbb{Z}/dp^N\mathbb{Z})$, $X^*=\bigcup_{0<a<dp} a+dp\mathbb{Z}_p$ and
$a+dp^N\mathbb{Z}_p=\{x\in X|x\equiv a \pmod{dp^N}\}$, where $a\in\mathbb{Z}$ with $0\leq a<dp^N$.
For any positive integer N,

$$\mu_q(x+dp^N\mathbb{Z}_p)=\frac{q^x}{[dp^N]}=\frac{q^x}{[dp^N:q]}$$

can be extended to a distribution on X,(cf.[5]).
Let $UD(\mathbb{Z}_p,\Omega_p)$ denote the space of all uniformly differentiable functions on $\mathbb{Z}_p$.
For $f\in UD(\mathbb{Z}_p,\Omega_p)$, this distribution yields an integral in the case $d=1$,

$$\int_{\mathbb{Z}_p} f(x)d\mu_q(x)=\lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)\frac{q^x}{[p^N]}=I_q(f)$$

, which has a sense as we see readily that the limit is convergent.
Recently, K.Dilcher has studied the formulas for sums of products of the form $\sum\binom{2n}{2j_1,\cdots,2j_N}B_{2j_1}\cdots B_{2j_N}$,(cf.[1]), and I.C.Huang also have studied the generalized formulas for sums of products of Bernoulli numbers,(cf.[2]). Later, T.Kim found formulas for sums of products of any number of Carlitz's q-Bernoulli numbers,(cf.[5]).
In this paper, we treat the some formulas to be related an invariant

p-adic integral on $\mathbb{Z}_p$ and give the formulas for sums of products of the analogue of Bernoulli numbers.

2. AN INVARIANT INTEGRAL ON $\mathbb{Z}_p$

By using an invariant integral on $\mathbb{Z}_p$, we consider the following numbers:

For $n \in \mathbb{Z}^+ = \{\text{the set of positive integers}\}$,

$$B_{n,q} = \int_{\mathbb{Z}_p} x^n d\mu_q(x).$$

Indeed, these numbers are analogue of Bernoulli numbers. So, we call these numbers an analogue of Bernoulli numbers.

Let $G(t)$ be the generating functions of the above analogue of Bernoulli numbers, that is , $G(t) = \sum_{n=0}^{\infty} \frac{B_{n,q}}{n!} t^n$.

Proposition 2.1. *For $q \in \Omega_p$ with $|1-q|_p < 1$, we have*

$$\mu_q(x) = q^x \mu_0(x),$$

where $\mu_0(x + p^N \mathbb{Z}_p) = \frac{1}{p^N}$.

Proof. It is not difficult to prove proposition2.1. □

By using proposition 2.1, we obtain the following:

Proposition 2.2. (1) *For $q \in \Omega_p$ with $|1-q|_p < 1$, we have*
$G(t) = \frac{t+\log q}{qe^t - 1} = \sum_{n=0}^{\infty} \frac{B_{n,q}}{n!} t^n.$
(2) *For $q \in \Omega_p$ with $|1-q|_p > 1$, we obtain*
$G(t) = \frac{q-1}{qe^t - 1} = \sum_{n=0}^{\infty} \frac{B_{n,q}}{n!} t^n.$

Proof. It is well known that

$\int_{\mathbb{Z}_p} f(x+1)d\mu_0(x) = \int_{\mathbb{Z}_p} f(x)d\mu_0(x) + f'(0)$, (*cf.*[3])

By the definition of the above analogue of Bernoulli numbers, $B_{n,q}$,

we easily see:

$B_{n,q} = \int_{\mathbb{Z}_p} x^n d\mu_q(x) = \int_{\mathbb{Z}_p} q^x x^n d\mu_0(x)$.

To prove (1), it is sufficient to show that $\lim_{x\to 0} \frac{q^x e^{xt}-1}{x} = t + \log q$.

Indeed,

$$\begin{aligned}
\lim_{x\to 0} \frac{q^x e^{xt}-1}{x} &= \lim_{x\to 0} \frac{1}{x}\{\sum_{n=0}^{x} \binom{x}{n}(q-1)^n e^{xt} - 1\} \\
&= \lim_{x\to 0} \frac{1}{x}\{e^{xt} - 1 + \sum_{n=1}^{x} \binom{x}{n}(q-1)^n e^{xt}\} \\
&= t + \sum_{n=1}^{x} \frac{1}{n}\binom{-1}{n-1}(q-1)^n \\
&= t + \log q.
\end{aligned}$$

The proof of (2) is trivial,(cf. [3], [10]). □

For each $q_j \in \Omega_p(j \in \mathbb{Z}^+)$, let μ_{q_j} be the p-adic distribution on $\mathbb{Z}_p$, and let $\mu_q = \prod_{1\le j\le r} \mu_{q_j}$ be the product measure on the product space ${\mathbb{Z}_p}^r = \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$.

Corollary 2.3. *For $q_j \in \Omega_p(^\forall j)$, we have*

$$\int_{\mathbb{Z}_p} exp(xt)d\mu_{q_j}(x) = \begin{cases} \frac{t+\log q_j}{q_j e^t - 1} & if\ |1-q_j|_p < 1, \\ \frac{q_j - 1}{q_j e^t - 1} & if\ |1-q_j|_p > 1. \end{cases}$$

Let $x = (x_1, x_2, \cdots, x_r)$ be variables on ${\mathbb{Z}_p}^r$, and let $t_1, t_2, \cdots, t_r$ be the p-adic variables with sufficiently small absolute values so that $\exp(x_1 t_1 + \cdots + x_r t_r)$ converges for any $(x_1, x_2, \cdots, x_r) \in {\mathbb{Z}_p}^r$.

By the property of μ_q, we can obtain the following :

Lemma 2.4. *For $r \in \mathbb{Z}^+$, we have*

$$\int_{\mathbb{Z}_p^r} \exp(x_1t_1 + \cdots + x_rt_r)d\mu_q(x) = \begin{cases} \prod_{1\le j\le r} \frac{t_j+\log q_j}{q_je^{t_j}-1} & \text{if } |1-q_j|_p < 1, \\ \prod_{1\le j\le r} \frac{q_j-1}{q_je^{t_j}-1} & \text{if } |1-q_j|_p > 1. \end{cases}$$

By Lemma 2.4, we obtain the following theorem:

Theorem 2.5. *Let $m_1,m_2,\cdots,m_r \in \mathbb{Z}^+$, $y = (y_1,y_2,\cdots,y_r) \in {\mathbb{Z}_p}^r$.*

(1) *For $q_j(^\forall j)$ with $|1-q_j| < 1$, $\int_{\mathbb{Z}_p{}^r} \prod_{l=1}^r y_l{}^{m_l} d\mu_q(y)$ is the coefficient of $\frac{t_1{}^{m_1}}{m_1!}\frac{t_2{}^{m_2}}{m_2!}\cdots\frac{t_r{}^{m_r}}{m_r!}$ in the Laurent expansion of $\prod_{1\le j\le r}\frac{t_j+\log q_j}{q_je^{t_j}-1}$.*

(2) *For $q_j(^\forall j)$ with $|1-q_j| > 1$, $\int_{\mathbb{Z}_p{}^r} \prod_{l=1}^r y_l{}^{m_l} d\mu_q(y)$ is the coefficient of $\frac{t_1{}^{m_1}}{m_1!}\frac{t_2{}^{m_2}}{m_2!}\cdots\frac{t_r{}^{m_r}}{m_r!}$ in the Laurent expansion of $\prod_{1\le j\le r}\frac{q_j-1}{q_je^{t_j}-1}$.*

Proof. Theorem2.5 is proved by Lemma2.4. □

The Theorem2.5 is very useful for doing study p-adic multiple gamma functions, and is an answer to a part of question in [7].

Corollary 2.6. *For each $c_i \in \mathbb{Z}^+$, we have*

$\int_{\mathbb{Z}_p{}^r} x_1{}^{m_1}\cdots x_r{}^{m_r} d\mu_q(x) = \lim_{n_1,\cdots,n_r\to\infty} \frac{\sum_{1\le j\le r}\sum_{0\le x_j\le c_jp^N}\prod_{j=1}^r q^{x_j}x_j{}^{m_j}}{[c_1p^{n_1}][c_2p^{n_2}]\cdots[c_rp^{n_r}]}$,

(see [7]).

In general, many mathematicians have studied the properties to be related Bernoulli numbers of high order , (see [1], [2], [5], [9], [10], [11]).

We would like to define an analogue of Bernoulli numbers of high order by using p-adic q-integral on $\mathbb{Z}_p$.

Definition 2.7. Define an analogue of Bernoulli numbers with order $k \in \mathbb{Z}^+$ as follows:

$$\sum_{n=0}^{\infty}\frac{B_{n,q}{}^{(k)}}{n!}t^n=\begin{cases}\left(\frac{t+\log q}{qe^t-1}\right)^k & \text{if } |1-q|_p<1,\\ \left(\frac{q-1}{qe^t-1}\right)^k & \text{if } |1-q|_p>1.\end{cases}$$

Recently, the sums of products of Bernoulli numbers of high order have been studied by I.C.Huang and K.Dicher, (cf. [1], [2]). In particular , we give the formulas for sums of products of the analogue of Bernoulli numbers of high order.

Corollary 2.8. *For $k\in\mathbb{Z}^+$, we obtain*

$B_{n,q}{}^{(k)}=\sum_{n=a_1+a_2+\cdots+a_k}\binom{n}{a_1,\cdots,a_k}B_{a_{1,q}}B_{a_{2,q}}\cdots B_{a_{k,q}}$,
where $\binom{n}{a_1,\cdots,a_k}$ is multinomial.

Remark. The above formula is the same result of I.C. Huang ([2]) and K. Dilcher ([1]), corresponding to the case $q=1$.

3. APPLICATIONS

For $u\in\Omega_p$ with $|1-u|_p\geq 1$, the Euler numbers are defined by $\frac{1-u}{e^t-u}=\sum_{n=0}^{\infty}\frac{H_n(u)}{n!}t^n$, (cf. [3], [10]).

Let $a\in\mathbb{Z}$ with $0\leq a\leq dp^n-1$, $n\geq 0$. Then the p-adic Euler measure on $\mathbb{Z}_p$ was defined by K.Shiratani as follows :

$$E_u(a+dp^n)=\frac{u^{dp^n-a}}{1-u^{dp^n}},\ \text{(cf.[10])}.$$

This measure yields a p-adic Eulerian integral :

$$\int_{\mathbb{Z}_p} x^n dE_u(x) = \lim_{n\to\infty} \sum_{x=0}^{p^n-1} x^n \frac{u^{p^n-x}}{1-u^{p^n}}.$$

Thus we have

$$\int_{\mathbb{Z}_p} x^n dE_u(x) = \frac{u}{1-u} H_n(u),$$

for $n \geq 0$, (cf. [10]).

For $q(\neq 1) \in \Omega_p$ with $|1-q|_p < 1$, we define the analogue Bernoulli numbers as follows :

$$\frac{\log q + t}{qe^t - 1} = \sum_{n=0}^{\infty} B_{n,q} \frac{t^n}{n!}.$$

Note that

$$\lim_{q\to 1} B_{n,q} = B_n,$$

where B_n are the ordinary Bernoulli numbers.

Let C_{p^n} be the cyclic group with order p^n and let $T_p = \varinjlim C_{p^n}$. Indeed, T_p is the set of local constant. For $q(\neq 1) \in T_p$, the analogue of Bernoulli numbers was defined in [3] as follows:

$$\frac{t}{qe^t - 1} = \sum_{n=0}^{\infty} \frac{\beta_n}{n!} t^n.$$

Thus our analogue of Bernoulli numbers have the following properties :

If $q \in \Omega_p$ with $|1-q|_p > 1$, then $B_{n,q} = H_n(q^{-1})$.

If $q \in T_p$, then $B_{n,q} = \beta_n$.

References

[1] K.Dilcher, *Sums of products of Bernoulli numbers*,J. Number Theory 60(1) (1996), 23-41.

[2] I.C.Huang, *Bernoulli numbers and polynomials via residue*, J. Number Theory 76 (1999), 178-193.

[3] T.Kim, *An analogue of Bernoulli numbers and their congruences*, Rep. Fac. Sci. Engrg. Saga Univ. Math. 22 (1994), 21-26.

[4] T.Kim, *On explicit formulars of p-adic q-L-functions*,Kyushu J. Math. 48(1994), 73-86.

[5] T.Kim, *Sums of products of q-Bernoulli numbers*,Arch.Math.76(2001), 190-195.

[6] T.Kim, *An invariant p-adic integral associated with Daehee numbers*, to appear in Integral Trans. Special Function (2001-2002).

[7] T.Kim, *Notes on multivariate p-adic q-integral (preprint).*

[8] T.Kim, *On p-adic $q-L$-functions and sums of powers*,to appear in Discrete Math.

[9] G.J.Murphy, *Translation-invariant Function Algebras on compact groups*, (to appear) in Advanced Stud. Contemp. Math. Vol. 3 No. 2 (2001).

[10] K.Shiratani, *On a p-adic interpolating function for the Euler number and it's derivatives*, Mem. Fac. Sci. Kyushu. Univ. Math. 39 (1985), 113-125.

[11] H.M.Srivastava, *An explicit formula for the generalized Bernoulli polynomials*, J. Math. Analysis and Application 130 (1988), 509-513.

DEPARTMENT OF MATHEMATICS, COLLEGE OF NATIONAL SCIENCES, KYUNGPOOK NATIONAL UNIVERSITY, TAEGU 702-701, KOREA

E-mail address: Haskcho@hanmail.net

E-mail address: Eskim@knu.ac.kr