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An Application of polylogarithms in the analogs of Genocchi numbers

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Abstract In this note, we will give a new formulae on Genocchi numbers. Also we define poly Genocchi numbers to give the relation between Genocchi number and poly Genocchi number .
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1. Introduction

The Genocchi numbers G_m are defined by the generating function

$$F(t)=\frac{2t}{e^t+1}=\sum_{m=0}^{\infty}G_m\frac{t^m}{m!},\quad (|x|<\pi). \tag{1}$$

It satisfies $G_1=1$, $G_3=G_5=G_7=\ldots=0$, and even coefficients are given by $G_m=2(1-2^{2m})B_{2m}=2nE_{2n-1}(0)$, where B_m are Bernoulli numbers and $E_n(x)$ are the Eulerian polynomials. The first few Genocchi numbers for n even are $-1,1,-3,17,-155,2073,\cdots$. By (1), we easily see that

$$G_0=0,$$

$$(G+1)^m+G_m=\begin{cases}2, & \text{if } k=1\\ 0, & \text{if } k>1,\end{cases}$$

with the usual convention of replacing G^m by G_m.

It follows from (1) and Von Stadut-Clausen theorem that the Genocchi numbers are integers. The following formulas ((2)-(3)) were proved by F.T.Howard (see [1]):

$$E_m(x)=\sum_{k=0}^{m} C_m^k \frac{G_{k+1}}{k+1} x^{m-k}, \text{where } E_m(x) \text{ are the Eulerian polynomials.} \tag{2}$$

For $m, n \geq 1$ and n odd

$$(n^m - n)G_m = \sum_{k=1}^{m-1} C_m^k n^k G_k Z_{m-k}(n-1), \tag{3}$$

where $Z_m(n) = 1^m - 2^m + 3^m - \cdots + (-1)^{n+1} n^m$.

In this note, we derive some formulae on Genocchi numbers and also we define poly Genocchi numbers to be related Genocchi numbers .

2. Some formulae on the Genocchi numbers

The polylogarithms were defined by

$$Li_k(x) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}, \quad |z| < 1, \quad k \in N = \{0, 1, 2, 3, \cdots\}, \text{ (cf. [2]).}$$

We define the Genocchi numbers of higher order as

$$\frac{2Li_k(1-e^{-x})}{e^x+1} = \sum_{n=0}^{\infty} G_n^{(k)} \frac{x^n}{n!}. \tag{4}$$

Note that $G_n^{(1)} = G_n$. For $x \in R$(= the field of real number), it was known that

$$\frac{d}{dx} Li_k(1-e^{-x}) = \frac{1}{e^x - 1} Li_{k-1}(1-e^{-x}), \quad \text{for} \quad k \geq 1. \tag{5}$$

By (4),(5), we see that

$$\frac{2}{e^x+1}\int_0^x \frac{1}{e^x-1} \underbrace{\int_0^t \frac{1}{e^t-1} \cdots \int_0^t \frac{t}{e^t-1}}_{k-1 \text{ times}} dt dt \cdots dt = \sum_{n=0}^{\infty} G_n^{(k)} \frac{x^n}{n!}. \tag{6}$$

Let $S(n,m)$ be the Stirling number of the second kind :

$$x^n = \sum_{m=0}^{n} S(n,m)(x)_m, \quad \text{where } (x)_m = x \cdot (x-1) \cdots (x-m+1),\ (x)_0 = 1.$$

Then they satisfy the following formulae (when $n = 0$, the identity $0^0 = 1$ understood):

$$S(n,m) = \frac{(-1)^m}{m!} \sum_{l=0}^{m} (-1)^l C_m^l l^n, \quad \frac{(e^t-1)^m}{m!} = \sum_{n=m}^{\infty} S(n,m) \frac{t^n}{n!}. \tag{7}$$

By (4), (7), we get

$$G_n^{(k)} = \sum_{l=0}^{n} G_{n-l} \sum_{m=0}^{l} C_n^l \frac{(-1)^{m+l}(m+1)!}{(m+1)^k} \frac{S(l+1,m+1)}{l+1}, \quad (n \geq 0, \forall k). \tag{8}$$

In (1), (4), (6), we see that

$$\sum_{n=0}^{\infty} G_n^{(2)} \frac{x^n}{n!} = \frac{1}{e^x+1} \int_0^x \frac{2t}{e^t+1} dt = \sum_{n=0}^{\infty} \{1/2 \sum_{l=0}^{n} C_n^l \frac{G_l G_{n-l}}{l+1}\} \frac{x^n}{n!}.$$

Hence, we can find the formula as follows:

$$2G_n^{(2)} = \sum_{l=0}^{n} C_n^l \frac{G_l Gn - l}{l+1}. \tag{9}$$

If n is odd number in (9), then $G_n^{(2)} = \frac{(n+2)}{4} G_{n-1}$.

For $s \in C$(= the complex number field) with Re$(s > 1)$, it is easy to see that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t+1} dt.$$

For $s \in C$ with Re$(s) > 1$, define $\zeta_G(s) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$.

Note that

$$\zeta_G(1-k) = \begin{cases} -1, & \text{if } k = 1 \\ -\frac{G_k}{k}, & \text{if } k > 1, \end{cases} \quad \text{for } k \in N . \tag{10}$$

We can consider the following functions in [2]:

$$\zeta_G^{(k)} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{2t^{s-1}}{e^t+1} Li_k(1-e^{-t}) dt, \quad \text{where } s \in C,\ k \in N. \tag{11}$$

It is easy to see that

$$\zeta_G^{(1)}(s) = s\zeta_G(s+1), \quad \text{and} \quad \zeta_G^{(k)}(-m) = (-1)^m G_m^{(k)}. \tag{12}$$

Now, we define the Genocchi polynomials as

$$\frac{2te^{xt}}{e^t+1} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}. \tag{13}$$

By (13), (1), we have

$$G_1(x+1) = G_1(x), \quad G_n(x+1) + G_n(x) = 2nx^{n-1}, \text{for } n = 2,4,5,\cdots. \tag{14}$$

Note that $G_n(1) = -G_n(0) = -G_n$, for $n \geq 2$.

By (13), we can find the following (15):

$$\sum_{n=0}^{\infty} G_n(x+1)\frac{t^n}{n!} = \frac{2te^{xt}e^t}{e^t+1} = \sum_{k=0}^{\infty} G_k(x)\frac{t^k}{k!}\sum_{l=0}^{\infty}\frac{t^l}{l!}. \tag{15}$$

By (13), (15), we see that

$$\sum_{k=0}^{n-1} C_n^k G_k(x) + 2G_n(x) = 2nx^{n-1}, \text{for } n \geq 1 \text{ and } \quad G_n(1-x) = (-1)^{n-1}G_n(x).$$

Differentiating both side with respect to x and comparing coefficients in (13), we have

$$G_{n-1}(x) = \frac{G_n'(x)}{n}. \tag{16}$$

The above (16) implies that $\int_x^y G_n(t)dt = \frac{G_{n+1}(y)-G_{n+1}(x)}{n+1}$.

Let $[x]$ be the largest integers $\leq x$, and $\{x\} = x - [x]$. Now, we define the nth Genocchi periodic function as $P_n(x) = G_n(\{x\})$, for $n \geq 1$. It is periodic with period 1 and agrees with the Genocchi polynomial $G_n(x)$ in the interval $0 \leq x < 1$. Let $P_m(x) = \sum_{n=-\infty}^{\infty} a_n^{(m)} e^{\pi inx}$, where $a_n^{(m)} \in C$, $n \neq 0, m \geq 1, i = (-1)^{\frac{1}{2}}$. By the Fourier series expansion, it is easy to see that

$$a_n^{(m)} = \int_0^1 P_m(x)e^{-\pi inx}dx = -(cos(n\pi)+1)\frac{G_{m+1}}{m+1} + \frac{\pi in}{m+1}a_n^{(m+1)}, \quad m \geq 1.$$

Thus

$$\begin{aligned} a_n^{(m+1)} &= \frac{m+1}{\pi in}[a_n^{(m)} + (1+cos(n\pi))\frac{G_{m+1}}{m+1}] \\ &= \frac{(m+1)m}{(\pi in)^2}a_n^{(m-1)} + (1+cos(n\pi))[\frac{(m+1)G_m}{(\pi in)^2} + \frac{G_{m+1}}{\pi in}] = \cdots \\ &= \frac{(m+1)!a_n^{(1)}}{(\pi in)^m} + (1+cos(n\pi))\sum_{j=1}^{m}\frac{G_{m-j+2}}{(\pi in)^j}\frac{(m+1)!}{(m-j+2)!}. \end{aligned}$$

Note that

$$a_n^{(1)} = \int_0^1 P_1(x)e^{-\pi inx}dx = \int_0^1 e^{-\pi inx}dx = \frac{(1-(-1)^n)}{\pi in},$$

and

$$a_0^{(m)} = \int_0^1 P_m(x)dx = \frac{1}{m+1}[P_{m+1}(x)]_0^1 = -\frac{2G_{m+1}}{m+1}, \text{ for } m \geq 1.$$

Hence

$$P_m(x) + \frac{2G_{m+1}}{m+1} = \sum_{n=-\infty, n\neq 0}^{\infty} [\frac{m!(1+(-1)^{n-1})}{(\pi in)^m} + (1+(-1)^n)\sum_{j=1}^{m-1}\frac{G_{m-j+1}}{(\pi in)^j}\frac{m!}{(m-j+1)!}]e^{\pi inx}.$$

Therefore we obtain the following: For $n \geq 1$ and $0 \leq x < 1$, we have

$$G_n(x) = \sum_{k=-\infty, k\neq 0}^{\infty} [\frac{n!(1+(-1)^{k-1})}{(\pi ik)^n} + (1+(-1)^k)\sum_{j=1}^{n-1}\frac{G_{n-j+1}}{(\pi ik)^j}\frac{n!}{(n-j+1)!}]e^{\pi ikx} - \frac{2G_{n+1}}{n+1}.$$

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