

A NOTE ON THE VALUES OF ZETA

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Abstract: By q -calculation, we construct q -analogue of Riemann ζ -function, Hurwitz's zeta function and prove some formulas for the values of these functions. By using this formulae, we can evaluate the values of $\zeta(5) - A\zeta(5, \frac{1}{4})$, where A is some rational number.

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Carlitz considered the numbers $\eta_k(q)$ which are determined by

$$\eta_0(q) = 1,$$

$$(q\eta(q) + 1)^k - \eta_k(q) = \begin{cases} 1, & \text{if } k = 1 \\ 0, & \text{if } k > 1 \end{cases}.$$

These numbers $\eta_k(q)$ induce Carlitz's k th q -Bernoulli numbers $\beta_k(q) = \beta_k$ as

$$\beta_0 = 1,$$

$$q(q\beta + 1)^k - \beta_k = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases},$$

where we use the usual convention about replacing β^i by β_i ($i \geq 0$).

Now, we modify the above number $\eta_m(q)$, that is,

$$B_0(q) = \frac{q-1}{\log q},$$

$$(qB(q) + 1)^k - B_k(q) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases},$$

where we use the usual convention about replacing $B^i(q)$ by $B_i(q)$ ($i \geq 0$) (see [2]).

In [2], we can constructe a complex q -series which is a q -analogue of Hurwitz's ζ -function. In this a short note, we will compute the values of zeta by using the q -series.

Let $F_q(t)$ be the generating function of $B_i(q)$:

$$F_q(t) = \sum_{k=0}^{\infty} B_k(q) \frac{t^k}{k!} \text{ for } q \in \mathcal{C} \text{ with } |q| < 1.$$

This is the unique solution of the following q -difference equation:

$$F_q(t) = e^t F_q(qt) - t.$$

It is easy to see that

$$F_q(t) = -t \sum_{n=0}^{\infty} q^n e^{[n]t} + \frac{q-1}{\log q} e^{\frac{1}{1-q}t}.$$

Thus we have:

$$B_k(q) = \left. \frac{d^k}{dt^k} F_q(t) \right|_{t=0} = -k \sum_{n=0}^{\infty} q^n [n]^{k-1} + \frac{(-1)^k}{(q-1)^{k-1} \log q}.$$

Hence, we can define a q -analogue of the ζ -function as follows.

For $s \in \mathcal{C}$, define

$$\zeta_q(s) = \sum_{n=1}^{\infty} \frac{q^n}{[n]^s} - \frac{1}{s-1} \frac{(1-q)^s}{\log q}.$$

Note that, $\zeta_q(s)$ is analytic continuation in \mathcal{C} with only one simple pole at $s = 1$ and

$$\zeta_q(1-k) = -\frac{B_k(q)}{k} \text{ where } k \text{ is any positive integer.}$$

Now, we define q -Bernoulli polynomial $B_n(x; q)$ as

$$B_n(x; q) = (q^x B(q) + [x])^n = \sum_{k=0}^n \binom{n}{k} q^{xi} B_k(q) [x]^{n-k}.$$

Let $T_q(x, t)$ be generating function of q -Bernoulli polynomials.

Note that

$$T_q(x, t) = F_q(q^x t) e^{[x]t}.$$

Thus

$$\begin{aligned} B_{k+1}(x; q) &= \left. \frac{d^{k+1}}{dt^{k+1}} T_q(x, t) \right|_{t=0} \\ &= -(k+1) \sum_{n=0}^{\infty} ([n]q^x + [x])^k q^{n+x} + \frac{q-1}{\log q} \left(\frac{1}{q-1} \right)^{k+1}. \end{aligned}$$

So, we can also define a q -analogue of the Hurwitz ζ -function as follows:

For $s \in \mathcal{C}$,

$$\begin{aligned} \zeta_q(s, x) &= \sum_{n=0}^{\infty} \frac{q^{n+x}}{([n]q^x + [x])^s} - \frac{(1-q)^s}{\log q} \frac{1}{s-1} \\ &= \sum_{n=0}^{\infty} \frac{q^{n+x}}{[n+x]^s} - \frac{(1-q)^s}{\log q} \frac{1}{s-1}, \quad 0 < x \leq 1. \end{aligned}$$

Note that, $\zeta_q(s, x)$ has an analytic continuation in \mathcal{C} with only one simple pole at $s = 1$.

Remark: $\zeta_q(s, x)$ is called q -analogue of Hurwitz ζ -function.

For $u \in \mathcal{C}$ with $u \neq 0, 1$, let $H_k(u : q)$ be q -Euler numbers (see [4]). It is known in [4] that $H_k(u : 1) = H_k(u)$ is the ordinary Euler number which is defined by

$$\frac{1-u}{e^t-u} = \sum_{k=0}^{\infty} H_k(u) \frac{t^k}{k!}.$$

In the case $u = -1$, $H_k(-1) = E_k$ is the classical Euler number which is defined by

$$\frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.$$

Note that $E_{2k} = 0$ ($k \geq 1$). In [4], $\ell_q(s, u)$ is defined by $\ell_q(s, u) = \sum_{n=1}^{\infty} \frac{u^{-n}}{[n]^s}$ and $\ell_q(-k, u) = \frac{u}{u-1} H_k(u : q)$ for $k > 1$.

By simple calculation, we can obtain the following (1)-(2):

For $s \in \mathcal{C}$, $f \in \mathcal{N} \setminus \{1\}$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{[n]^s} q^n = -\zeta_q^*(s) + \frac{2}{[2]^s} \zeta_{q^2}^*(s).$$

$$\zeta_q^*(s) = \frac{1}{[f]^s} \sum_{a=1}^f \zeta_{q^f}^*(s, \frac{a}{f}), \text{ where } \zeta_q^*(s) = \sum_{n=1}^{\infty} \frac{q^n}{[n]^s}.$$

It is easy to see that

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{[n]^{2k+1}} \sum_{j=0}^{\infty} \frac{\theta^{2j+1} [n]^{2j+1}}{(2j+1)!} + \frac{1}{\log q} \sum_{j=0}^{k-1} \frac{(1-q)^{2k-2j}}{2k-2j-1} \frac{\theta^{2j+1}}{(2j+1)!} \\ & - \frac{1}{\log q} \sum_{j=0}^{k-1} \frac{\theta^{2j+1}}{(2j+1)!} \frac{(1-q^2)^{2k-2j}}{2k-2j-1} \frac{1}{[2]^{2k-2j}} \\ & = \sum_{j=0}^{k-1} \frac{(-1)^j \theta^{2j+1}}{(2j+1)!} \left(-\zeta_q(2k-2j) + \zeta_{q^2}(2k-2j) \frac{2}{[2]^{2k-2j}} \right) \\ & - \frac{q}{1+q} \frac{\theta^{2k+1}}{(2k+1)!} (-1)^k + \sum_{j=k+1}^{\infty} \frac{\theta^{2j+1} (-1)^j}{(2j+1)!} \frac{q^{-1}}{1+q^{-1}} H_{2j-2k}(-q^{-1}, q). \end{aligned}$$

If $q \rightarrow 1$, then we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2k+1}} \sin(n\theta) \\ & = \sum_{j=0}^{k-1} \frac{(-1)^j}{(2j+1)!} \theta^{2j+1} \left(\frac{2}{2^{2k-2j}} - 1 \right) (-1)^{k-j+1} \frac{(2\pi)^{2k-2j}}{2 \cdot (2k-2j)!} B_{2k-2j} - \frac{1}{2} \frac{\theta^{2k+1}}{(2k+1)!} (-1)^k. \end{aligned}$$

Let $k = 2$ and $\theta = \frac{\pi}{2}$. Then we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^5} = \pi^5 \left(\frac{1}{2^6 \cdot 5!} - \frac{B_2}{2^4 \cdot 3!} + \frac{7}{12} B_4 \right).$$

It is easy to see that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^5} &= 2 \sum_{n=1}^{\infty} \frac{1}{(4n-3)^5} + \sum_{n=1}^{\infty} \frac{1}{(2n)^5} - \sum_{n=1}^{\infty} \frac{1}{n^5} - 1 \\ &= \frac{1}{2^9} \zeta\left(5, \frac{1}{4}\right) - \frac{2^5-1}{2^5} \zeta(5) - 1. \end{aligned}$$

Thus we have

$$\zeta(5) - \frac{1}{2^4} \frac{1}{2^5-1} \zeta\left(5, \frac{1}{4}\right) - 1 = -2^5 \pi^5 \left(\frac{1}{2^6 \cdot 5!} - \frac{B_2}{2^4 \cdot 3!} + \frac{7}{12} B_4 \right).$$

Therefore we obtain the following:

$$\zeta(5) - \frac{1}{2^4(2^5-1)} \zeta\left(5, \frac{1}{4}\right) - 1 \text{ is irrational.}$$

References

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