NNTDM 7 (2001) 1, 15-20

THREE FORMULAE FOR n-th PRIME AND SIX FOR n-th TERM OF TWIN PRIMES

Mladen Vassilev - Missana
5, V. Hugo Str., Sofia–1124, Bulgaria

Abstract: Let $C = \{C_n\}_{n\geq 1}$ be an arbitrary increasing sequence of natural numbers. By $\pi_C(n)$ we denote the number of the terms of C being not greater than n (we agree that $\pi_C(0) = 0$). In the first part of the paper we propose six different formulae for C_n ($n = 1, 2, ...$), which depend on the numbers $\pi_C(k)$ ($k = 0, 1, 2, ...$). Using these formulae, in the second part of the paper we obtain three different explicit formulae for the n-th prime p_n, which are the first main result of the present research. In the third part of the paper, using the formulae from the first part, we propose six explicit formulae for the n-th term of the sequence of twin primes: 3,5,7,11,13,17,19,... - the second main result of the paper. The last three of them are main ones for the twin primes.

Used denotations

$[x]$ is used for the largest integer not greater than the real nonnegative number x; $\pi(k)$ - for the prime counting function; $\pi_2(k)$ - for the twin primes counting function, i.e., $\pi_2(n)$ denotes the number of primes p such that $p \leq n$ and $p+2$ are also primes; $p_2(n)$ - for n-th term of the twin primes sequence (for example, $p_2(1) = 3, p_2(2) = 5, p_2(3) = 7, ...$), ζ - for the Riemann's function zeta; Γ - for the Euler's function gamma; φ - for Euler's function, $\psi(n)$ - for Dedekind's function, $\sigma(n)$ - for the sum of all divisors, i.e., $\varphi(1) = \psi(1) = \sigma(1) = 1$ and

$$\varphi(n) = n.\prod_{i=1}^{k}(1 - \frac{1}{p_i}); \quad \psi(n) = n.\prod_{i=1}^{k}(1 + \frac{1}{p_i}); \quad \sigma(n) = \prod_{i=1}^{k}\frac{p_i^{\alpha_i+1} - 1}{p_i - 1}.$$

where $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ is a pime number factorization of n.

Part 1: UNIVERSAL FORMULAE FOR THE n-th TERM OF AN ARBITRARY INCREASING SEQUENCE OF NATURAL NUMBERS

1. A bracket function formula for C_n:

$$C_n = \sum_{k=0}^{\infty}[\frac{1}{1 + [\frac{\pi_C(k)}{n}]}]. \tag{1}$$

2. A formula using Riemann's function ζ:

$$C_n = -2.\sum_{k=0}^{\infty}\zeta(-2.[\frac{\pi_C(k)}{n}]). \tag{2}$$

3. A formula using Euler's function Γ:

$$C_n = \sum_{k=0}^{\infty} \frac{1}{\Gamma(1 - [\frac{\pi_C(k)}{n}])}. \tag{3}$$

Proof of the formulae (1)-(3). First, we represent (2) in the form

$$C_n = \sum_{k=0}^{\infty} (-2).\zeta(-2.[\frac{\pi_C(k)}{n}]). \tag{2'}$$

After that for each one of (1), (2′), (3) we use that

$$\sum_{k=0}^{\infty} \bullet = \sum_{k=0}^{C_n - 1} \bullet + \sum_{k=C_n}^{\infty} \bullet.$$

Let $k = 0, 1, ..., C_n - 1$. Then we have $\pi_C(k) \leq \pi_C(C_n - 1) < \pi_C(C_n) = n$. Hence $[\frac{\pi_C(k)}{n}] = 0$ for $k = 0, 1, ..., C_n - 1$. Therefore, for (1) we have

$$\sum_{k=0}^{C_n-1} [\frac{1}{1 + [\frac{\pi_C(k)}{n}]}] = \sum_{k=0}^{C_n-1} 1 = C_n.$$

The same way, for (2′) we have

$$\sum_{k=0}^{C_n-1} (-2)\zeta(-2.[\frac{\pi_C(k)}{n}]) = \sum_{k=0}^{C_n-1} (-2)\zeta(0) = \sum_{k=0}^{C_n-1} 1 = C_n,$$

since it is known that $\zeta(0) = -\frac{1}{2}$ (see [1]).

About (3) we have

$$\sum_{k=0}^{C_n-1} \frac{1}{\Gamma(1 - [\frac{\pi_C(k)}{n}])} = \sum_{k=0}^{C_n-1} \frac{1}{\Gamma(1)} = \sum_{k=0}^{C_n-1} 1 = C_n.$$

Let $k = C_n, C_n + 1, C_n + 2, ...$ Then we have $n = \pi_C(C_n) \leq \pi(k)$.

Therefore, $[\frac{\pi_C(k)}{n}] \geq 1$ for $k = C_n, C_n + 1, C_n + 2, ...$ Hence:

$$[\frac{1}{1 + [\frac{\pi_C(k)}{n}]}] = 0$$

for $k = C_n, C_n + 1, C_n + 2, ...$ Therefore, for (1) $\sum\limits_{k=C_n}^{\infty}$ vanishes. This proves (1).

To prove (2′) (i.e., (2)) it remains to show that $\sum\limits_{k=C_n}^{\infty}$ vanishes as in the previous case. But this is obvious from the fact that for $k = C_n, C_n + 1, C_n + 2, ...$

$$n_k \equiv [\frac{\pi_C(k)}{n}]$$

is a natural number and therefore

$$\zeta(-2n_k) = 0.$$

Since, the negative even numbers are trivial zeros of Riemann's function ζ (see [1]). We also have

$$\frac{1}{\Gamma(1-n_k)} = 0$$

for $k = C_n, C_n+1, C_n+2, \ldots$. Since, it is known that the nonpositive integers are poles of Euler's function gamma. Therefore, for (3) the sum $\sum\limits_{k=C_n}^{\infty}$ vanishes too, which proves (3).

4. Three other formulae for $\mathbf{C_n}$:

$$C_n = \sum_{k=0}^{\infty}[\frac{1}{1+[\frac{\pi_C(k)+n}{2n}]}]. \tag{1*}$$

$$C_n = -2.\sum_{k=0}^{\infty}\zeta(-2.[\frac{\pi_C(k)+n}{2n}]). \tag{2*}$$

$$C_n = \sum_{k=0}^{\infty}\frac{1}{\Gamma(1-[\frac{\pi_C(k)+n}{2n}])}. \tag{3*}$$

The validity of these formulae are checked analogically.

Part 2: FORMULAE FOR n-th PRIME $\mathbf{p_n}$

Here as a corollary from Part 1 we propose three formulae for p_n, which are finite. Let

$$\theta(n) = [\frac{n^2+3n+4}{4}].$$

It is known (see [2]) that

$$p_n \le \theta(n)$$

for $n = 1, 2, \ldots$. Hence

$$p_n < n^2$$

for $n > 1$. Then, if we put

$$C_n = p_n$$

for $n = 1, 2, \ldots$ and using that

$$\pi_C(n) = \pi(n),$$

we obtain the following formulae from (1), (2) and (3):

$$\mathbf{p_n} = \sum_{\mathbf{k=0}}^{\theta(\mathbf{n})}[\frac{\mathbf{1}}{\mathbf{1}+[\frac{\pi(\mathbf{k})}{\mathbf{n}}]}]; \tag{4}$$

$$\mathbf{p_n} = -\mathbf{2}.\sum_{\mathbf{k=0}}^{\theta(\mathbf{n})}\zeta(-\mathbf{2}.[\frac{\pi(\mathbf{k})}{\mathbf{n}}]); \tag{5}$$

$$\mathbf{p_n} = \sum_{\mathbf{k=0}}^{\theta(\mathbf{n})} \frac{\mathbf{1}}{\mathbf{\Gamma(1-[\frac{\pi(k)}{n}])}}. \tag{6}$$

The above formulae stay valid if we change $\theta(n)$ with n^2. These formulae are explicit ones, because $\pi(k)$ has explicit representations (see [3,5]).

One may compare (4) with the formula of Willans (see [3]):

$$p_n = 1 + \sum_{k=1}^{2^n} [[\frac{n}{1+\pi(k)}]^{\frac{1}{n}}].$$

Part 3: FORMULAE FOR $\mathbf{p_2(n)}$

Let $C_n = p_2(n)$. In this case we have

$$\pi_C(0) = \pi_C(1) = \pi_C(2) = 0;\ \pi_C(3) = \pi_C(4) = 1. \tag{*}$$

When $k \geq 5$ it is easy to see that

$$\pi_C(k) = \begin{cases} 2\pi_2(k) - 2, & \text{if } k-1 \text{ and } k+1, \text{ or } k \text{ and } k+2 \text{ are twin primes} \\ 2\pi_2(k) - 1, & \text{otherwise} \end{cases}, \tag{6'}$$

or in an explicit form

$$\pi_C(k) = 2\pi_2(k) - 1 - \delta(k-1) - \delta(k), \tag{6''}$$

where

$$\delta(k) = \begin{cases} 1, & \text{if } k \text{ and } k+2 \text{ are twin primes} \\ 0, & \text{otherwise} \end{cases}.$$

It is easy to give an explicit representation of $\delta(k)$:

$$\delta(k) = [\frac{2(k-1)! + (k+1)! + 2}{k(k+2)} - [\frac{2(k-1)! + (k+1)!}{k(k+2)}]]. \tag{6'''}$$

It is possibe to use instead of (6″) the representation:

$$\pi_C(k) = \pi_2(k) + \pi_2(k-2) - 1,$$

since $\pi_2(k) = \sum_{j=3}^{k} \delta(j)$.

Therefore, from (1) – (3) we obtain the corresponding formulae for $p_2(n)$:

$$p_2(n) = \sum_{k=0}^{\infty} [\frac{1}{1 + [\frac{\pi_C(k)}{n}]}]; \tag{7}$$

$$p_2(n) = -2. \sum_{k=0}^{\infty} \zeta(-2.[\frac{\pi_C(k)}{n}]); \tag{8}$$

$$p_2(n) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(1 - [\frac{\pi_C(k)}{n}])}, \tag{9}$$

where $\pi_C(k)$ is given by (*) for $k = 0, 1, 2, 3, 4$, and by (6'') for $k \geq 5$ with $\delta(k)$ is given by (6''').

Three new explicit formulae for $p_2(n)$ for even $n > 2$ are given below, while $p_2(2) = 5$. They correspond to (1*) – (3*) and use (6'):

$$\mathbf{p_2(n) = 5 + \sum_{k=5}^{\infty} [\frac{1}{1 + [\frac{\pi_2(k) - 1 + \frac{n}{2}}{n}]}];} \tag{7*}$$

$$\mathbf{p_2(n) = 5 - 2. \sum_{k=5}^{\infty} \zeta(-2.[\frac{\pi_2(k) - 1 + \frac{n}{2}}{n}]);} \tag{8*}$$

$$\mathbf{p_2(n) = 5 + \sum_{k=5}^{\infty} \frac{1}{\Gamma(1 - [\frac{\pi_2(k) - 1 + \frac{n}{2}}{n}])},} \tag{9*}$$

They follow from identity

$$[\frac{\pi_C(k) + n}{2n}] = [\frac{\pi_2(k) - 1 + \frac{n}{2}}{n}],$$

since for $k \geq 5$ $\pi_C(k)$ is given by (6') and for even $n > 2$ $[\frac{n-1}{2}] = \frac{n}{2} - 1$. Obviously, $p_2(1) = 3$, $p_2(3) = 7$ and for odd $n \geq 5$ we have $p_2(n) = p_2(n-1) + 2$ and may apply the formulae (7*) – (9*) for $p_2(n-1)$ since $n - 1$ is an even number.

The last three formulae are main ones for the twin primes.

All formulae for $p_2(n)$ are explicit, because in [4] some new explicit formulae for $\pi_2(n)$ are proposed. One of them is valid for $n \geq 5$:

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]} [\frac{2(6k-2)! + (6k)! + 2}{36k^2 - 1} - [\frac{2(6k-2)! + (6k)!}{36k^2 - 1}]].$$

For $\pi(n)$ one may use Mináč's formula (see [3]):

$$\pi(n) = \sum_{k=2}^{n} [\frac{(k-1)! + 1}{k} - [\frac{(k-1)!}{k}]],$$

or any of the following formulae, proposed here:

$$\pi(n) = -2. \sum_{k=2}^{n} \zeta(-2.(k - 1 - \varphi(k)); \tag{10}$$

$$\pi(n) = -2. \sum_{k=2}^{n} \zeta(-2.(\sigma(k) - k - 1); \tag{11}$$

$$\pi(n) = \sum_{k=2}^{n} [\frac{\varphi(k)}{k-1}]; \tag{12}$$

$$\pi(n) = \sum_{k=2}^{n} [\frac{k+1}{\sigma(k)}]; \tag{13}$$

$$\pi(n) = \sum_{k=2}^{n} [\frac{1}{k - \varphi(k)}]; \tag{14}$$

$$\pi(n) = \sum_{k=2}^{n} [\frac{1}{\sigma(k) - k}]. \tag{15}$$

Remark: In (11), (13), (15) one may prefer to put $\psi(k)$ instead of $\sigma(k)$ and then the formulae will remain valid.

The author expresses his deep gratitude to Krassimir Atanassov for his having inspired the present research by the following formulae, obtained by him, published in [5]:

$$\pi(n) = \sum_{k=2}^{n} \overline{sg}(k - 1 - \varphi(k));$$

$$\pi(n) = \sum_{k=2}^{n} \overline{sg}(\sigma(k) - k - 1);$$

$$\pi(n) = \sum_{k=2}^{n} fr(\frac{k}{(k-1)!}),$$

$$p_n = \sum_{i=0}^{2^n} sg(n - \pi(i)),$$

where:

$$sg(x) = \begin{cases} 0, & \text{if } x \le 0 \\ 1, & \text{if } x > 0 \end{cases}, \quad \overline{sg}(x) = \begin{cases} 0, & \text{if } x \ne 0 \\ 1, & \text{if } x = 0 \end{cases},$$

where x is a real number and

$$fr(\frac{p}{q}) = \begin{cases} 0, & \text{if } p = 1 \\ 1, & \text{if } p \ne 1 \end{cases},$$

where p and q are natural numbers, such that $(p, q) = 1$.

References

[1] Davenport, H. Multiplicative Number Theory. Markham Publ. Co., Chicago, 1967.

[2] Mitrinović, D., M. Popadić. Inequalities in Number Theory. Niś, Univ. of Niś, 1978.

[3] Ribenboim, P. The New Book of Prime Number Records. Springer, New York, 1995.

[4] Vassilev–Missana, M. Some new formulae for the twin primes counting function $\pi_2(n)$. Notes on Number Theory and Discrete Mathematics, Vol. 7, 2001, No. 1, 10-14.

[5] Atanassov, K. A new formula for the n-th prime number. Comptes Rendus de l'Academie Bulgare des Sciences, Vol. 54, No. 7, 5-6.