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SOME NEW FORMULAE FOR THE TWIN PRIMES COUNTING FUNCTION π_2(n)

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Abstract For every $n > 1$, let $\pi_2(n)$ denote the number of primes p such that $p \leq n$ and $p+2$ is also a prime. In the present paper some new formulae for $\pi_2(n)$ are proposed.

§0. Some denotations which are used

$\mathcal{N}$ - the set of all natural numbers (i.e., the set of all positive integers); $[x]$ - the greatest integer which is not greater than the real positive number x; $\zeta(s)$ - Riemann's zeta function; $\varphi(n)$ - Euler's function, $\psi(n)$ - Dedekind's function, $\sigma(n)$ - the sum of all divisors of n.

In particular: $\varphi(1) = \psi(1) = \sigma(1) = 1$ and if

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}$$

is the prime number factorization of n, then

$$\varphi(n) = n.\prod_{i=1}^{k}(1 - \frac{1}{p_i});$$

$$\psi(n) = n.\prod_{i=1}^{k}(1 + \frac{1}{p_i});$$

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1}.$$

§1. A bracket function formula for $\pi_2(n)$ using factorial

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]} [\frac{2(6k-2)! + (6k)! + 2}{36k^2 - 1} - [\frac{2(6k-2)! + (6k)!}{36k^2 - 1}]]. \qquad (1)$$

Here, and furthermore, $n \geq 5$ and $\pi_2(0) = \pi_2(1) = \pi_2(2) = 0; \pi_2(3) = 1$.

§2. Formulae for $\pi_2(n)$ using Riemann's zeta function

$$\pi_2(n) = 1 - 2.\sum_{k=1}^{[\frac{n+1}{6}]} \zeta(\varphi(6k-1) + \varphi(6k+1) - 12k + 2); \qquad (2)$$

$$\pi_2(n) = 1 - 2.\sum_{k=1}^{[\frac{n+1}{6}]} \zeta(12k + 2 - \psi(6k-1) - \psi(6k+1)); \quad (3)$$

$$\pi_2(n) = 1 - 2.\sum_{k=1}^{[\frac{n+1}{6}]} \zeta(24k + 4 - 2\sigma(6k-1) - 2\sigma(6k+1)). \quad (4)$$

§3. Bracket function formulae for $\pi_2(n)$ using Euler's function φ

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]} [\frac{\varphi(36k^2-1)}{36k^2-12k}]; \quad (5)$$

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]} [\frac{1}{2}.\sqrt{\frac{\varphi(36k^2-1)}{3k(3k-1)}}]; \quad (6)$$

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]} [\frac{\varphi(6k-1)+\varphi(6k+1)}{12k-2}]; \quad (7)$$

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]} [\frac{\varphi(6k-1)}{12k-4} + \frac{\varphi(6k+1)}{12k}]; \quad (8)$$

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]} [\frac{1}{6k - \frac{\varphi(6k-1)+\varphi(6k+1)}{2}}]. \quad (9)$$

§4. Bracket function formula for $\pi_2(n)$ using Dedekind's function ψ

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]} [\frac{36k^2+12k}{\psi(36k^2-1)}]; \quad (10)$$

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]} [2.\sqrt{\frac{3k(3k+1)}{\psi(36k^2-1)}}]; \quad (11)$$

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]} [\frac{12k+2}{\psi(6k-1)+\psi(6k+1)}]; \quad (12)$$

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]} [\frac{3k}{\psi(6k-1)} + \frac{3k+1}{\psi(6k+1)}]; \quad (13)$$

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]} [\frac{1}{\frac{\psi(6k-1)+\psi(6k+1)}{2} - 6k}]. \tag{14}$$

Remark: The formulae from §4 are still true if we put $\sigma(n)$ instead of $\psi(n)$.

§5. Proofs of the formulae

In order to prove all of the above formulae we need the arithmetic function

$$\delta(n) = \begin{cases} 1, & \text{if } k \text{ and } k+2 \text{ are twin primes} \\ 0, & \text{otherwise} \end{cases} \tag{15}$$

Since $p = 6k - 1$ if p and $p + 2$ are twin primes, we obtain for $n \geq 5$:

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]} \delta(6k-1). \tag{16}$$

First, let us prove (1). It is enough to prove that for $k \geq 5$ the equality

$$\delta(k) = [\frac{2(k-1)! + (k+1)! + 2}{k(k+2)} - [\frac{2(k-1)! + (k+1)!}{k(k+2)}]] \tag{17}$$

holds.

We rewrite (17) in the form

$$\delta(k) = [\frac{(k-1)!+1}{k} + \frac{k!-1}{k+2} - [\frac{(k-1)!}{k} + \frac{k!}{k+2}]]. \tag{18}$$

Further, we use a variant of Wilson's theorem given be Coblyn in 1913 (see [1]): *"The integer $m \geq 2$ is a prime iff m divides each of the numbers $(r-1)!(m-r)! + (-1)^{r-1}$ for $r = 1, 2, ..., m-1$."* The cases $r = 1$ and $r = 2$ are called Wilson's and Leibnitz theorem respectively [2]. We note by $g(k)$ the right hand-side of (18).

($\mathbf{a_1}$) Let k and $k+2$ be twin primes. Therefore, $(k-1)! + 1 = k.x$ ($x \in \mathcal{N}$) from the Wilson's theorem and $k! - 1 = ((k+2)-2)! - 1 = (k+2).y$ ($y \in \mathcal{N}$) from the Leibnitz's theorem. Hence:

$$g(k) = [\frac{kx}{k} + \frac{(k+2)y}{k+2} - [\frac{kx-1}{k} + \frac{(k+2)y+1}{k+2}]]$$

$$= [x + y - [x + y - (\frac{1}{k} - \frac{1}{k+2})]] = [x + y - (x+y-1)] = 1.$$

($\mathbf{a_2}$) Let k be prime and $k+2$ be composite. Therefore, $k > 6$. Now, it is easy to see that $k! = (k+2).y$ ($y \in \mathcal{N}$). The Wilson's theorem yields $(k-1)! + 1 = k.x$ ($x \in \mathcal{N}$). Hence:

$$g(k) = [\frac{kx}{k} + \frac{(k+2)y-1}{k+2} - [\frac{kx-1}{k} + \frac{(k+2)y}{k+2}]]$$

$$= [x + y - \frac{1}{k+2} - [x + y - \frac{1}{k}]] = [x + y - \frac{1}{k+2} - (x + y - 1)] = [1 - \frac{1}{k+2}] = 0.$$

($\mathbf{a_3}$) Let k be composite and $k+2$ be prime. Therefore, $k > 6$. Now, it is easy to see that $(k-1)! = k.x$ ($x \in \mathcal{N}$). The Leibnitz's theorem yields $k! - 1 = (k+2).y$ ($y \in \mathcal{N}$). Hence:

$$g(k) = [\frac{kx+1}{k} + \frac{(k+2)y}{k+2} - [\frac{kx}{k} + \frac{(k+2)y+1}{k+2}]]$$

$$= [x + \frac{1}{k} + y - [x + y + \frac{1}{k+2}]] = [x + y + \frac{1}{k} - (x+y)] = [\frac{1}{k}] = 0.$$

($\mathbf{a_4}$) Let k and $k+2$ be composite. Therefore, $k \geq 6$. Now, it is easy to see that $(k-1)! = k.x$ ($x \in \mathcal{N}$) and $k! = (k+2).y$ ($y \in \mathcal{N}$). Hence:

$$g(k) = [\frac{kx+1}{k} + \frac{(k+2)y-1}{k+2} - [\frac{kx}{k} + \frac{(k+2)y}{k+2}]]$$

$$= [x + y + \frac{1}{k} - \frac{1}{k+2} - (x+y)] = [\frac{1}{k} - \frac{1}{k+2}] = 0.$$

From (a_1) - (a_4) it follows that $g(k) = \delta(k)$ for $k \geq 5$ and the proof of (1) is finished.

Second, let us prove the formulae from §2. We need the well known fact that $\zeta(0) = -\frac{1}{2}$ and $\zeta(-2m) = 0$ for $m \in \mathcal{N}$ (see [3]). Since the numbers $\varphi(6k-1), \varphi(6k+1), \psi(6k-1), \psi(6k+1), 2\sigma(6k-1), 2\sigma(6k+1)$ are even, under the inequalities:

$$\varphi(6k-1) + \varphi(6k+1) \leq 12k - 2$$

$$\psi(6k-1) + \psi(6k+1) \geq 12k + 2$$

$$\sigma(6k-1) + \sigma(6k+1) \geq 12k + 2$$

and the fact that the last inequalities become equalities simultaneously iff $6k-1$ and $6k+1$ are twin primes, we conclude that the argument of the function ζ in (2) - (4) is everywhere nonpositive even number. Moreover, this argument equals to zero iff $6k-1$ and $6k+1$ are twin primes. Therefore, we have

$$\delta(6k-1) = -2\zeta(\varphi(6k-1) + \varphi(6k+1) - 12k + 2)$$

$$= -2\zeta(12k + 2 - \psi(6k-1) - \psi(6k+1)) = -2\zeta(24k + 4 - 2\sigma(6k-1) - 2\sigma(6k+1)).$$

Hence, (2) - (4) are proved because of (16).

It remains only to prove the formulae from §3 and §4.

First, we use that

$$\varphi(36k^2 - 1) = \varphi(6k-1).\varphi(6k+1)$$

and

$$\psi(36k^2 - 1) = \psi(6k-1).\psi(6k+1),$$

since, the functions φ and ψ are multiplicative.

Second, we use that inequalities $\varphi(6k-1) \leq 6k-2$ and $\varphi(6k+1) \leq 6k$ (just as inequalities $\psi(6k-1) \geq 6k$ and $\psi(6k+1) \geq 6k+2$) become equalities simultaneously iff numbers $6k-1$ and $6k+1$ are twin primes.

Then it is easy to verify that each one of the expressions behind sum $\sum_{k=1}^{[\frac{n+1}{6}]}$ in (5) - (14) equals to $\delta(6k-1)$. Hence, the proof of the formulae from §3 and 4 falls from (15).

References

[1] Ribenboim, P. The New Book of Prime Number Records. Springer, New York, 1995.
[2] Davenport, H. Multiplicative Numebr Theory. Markham Publ. Co., Chicago, 1967.
[3] Sierpinski, W. Co wiemy, a czego nie wiemy o liczbach pierwszych. Panstwowe Zaklady Wydawnictw Szkolnych Warszawa, 1961.