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ON ARITHMETIC FUNCTIONS AND A TRIGONOMETRICAL PRODUCT

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In what follows we shall study certain arithmetic functions with application to the study of some trigonometrical products.

1. The following auxilary propositions will be used. The first Lemma is well known.

LEMMA 1 (Möbius inversion formula) If $\sum_{d|n} f(d) = g(n)$, then

$$f(n) = \sum_{d|n} \mu(d) g(\frac{n}{d}),$$

where μ is the Möbius function.

Note: The sum $\sum_{d|n} a(d)b(\frac{n}{d})$, will be convolution $(a * b)(n)$. In Lemma 1, $f = \mu * g$.

LEMMA 2 (Hurwitz [8]) Let $\phi : [0,1] \to \mathcal{C}$ be an arbitrary complex-valued function and put $f(n) = \sum_{k \in A(n)} \psi(\frac{k}{n})$, $g(n) = \sum_{k=1}^{n} \psi(\frac{k}{n})$, where $A(n) = \{k | 1 \leq k \leq n \& (k,n) = 1\}$. Then

$$f(n) = \sum_{d|n} \mu(d) g(\frac{n}{d}) = (\mu * g)(n).$$

Proof: $g(n) = \sum_{d|n} \sum_{(k,n)=d} \psi(\frac{n}{d}) = \sum_{d|n} \sum_{s \in A(\frac{n}{d})} \psi(\frac{s}{\frac{n}{d}}) = \sum_{d|n} f(d).$

LEMMA 3: Let $\psi : [0,1] \to (0, \infty)$ and $F(n) = \prod_{k \in A(n)} \psi(\frac{k}{n})$, $G(n) = \prod_{k=1}^{n} \psi(\frac{k}{n})$. Then

$$F(n) = \prod_{d|n} [G(\frac{n}{d})]^{\mu(d)}$$

Proof: Since

$$\log F(n) = \log \prod_{k \in A(n)} \psi(\frac{k}{n}) = \sum_{k \in A(n)} \log \psi(\frac{k}{n})$$

and

$$\log G(n) = \sum_{k=1}^{n} \log \psi(\frac{k}{n}),$$

we can apply Lemma 2 with $f = \log F$, $g = \log G$. One gets

$$\log F(n) = \sum_{d|n} \mu(d) \log G(\frac{n}{d}) = \log \prod_{d|n} [G(\frac{n}{d})]^{\mu(d)}.$$

DEFINITION 1. Let λ be a given arithmetical function. If the prime factorization of $n > 1$ is given by $n = \sum_{i=1}^{\omega(n)} p_i^{a_i}$ (where $\omega(n)$ denotes the number of distinct prime divisors of n), let

$$u(n) = \begin{cases} \lambda(p), & \text{if } \omega(n) = 1 \\ 0, & \text{otherwise} \end{cases}$$

and $v(n) = \sum_{i=1}^{\omega(n)} a_i \lambda(p_i)$.

Further, let $u(1) = v(1) = 1$.

This function v has been named by S. W. Golomb [3] a logarithmic function.

We now can prove the following

LEMMA 4: The above introduced function v is completely additive, i.e., $v(mn) = v(m) + v(n)$ for all $m, n \geq 1$.

We omit the simple proof of this result.

CONSEQUENCE: $v(n^k) = k.v(n)$ for $n, k \geq 1$.

LEMMA 5: We have $v(n) = \sum_{d|n} u(d)$ and $u(n) = (\mu * v)(n)$.

Proof: By using the definition of the function u, we must consider only those divisors of n which have a single prime factor. Since $p_i|n$, $p_i^2|n$, ..., $p_i^{a_i}|n$ and always $u(p_i^s) = \lambda(p_i)$ and this term appears a_i times in the sum $(1 \leq s \leq a_i)$, the identity follows. The second equality follows by Lemma 1.

LEMMA 6 ([9], p. 104) Let f, g, h be arbitrary arithmetic functions, where h is completely additive. Then $(f * g).h = (fh) * g + f * (gh)$.

Proof: $(f * g)(n).h(n) = \sum_{d|n} f(d)g(\frac{n}{d})h(n) = \sum_{d|n} (f(d)g(\frac{n}{d})h(d) + f(d)g(\frac{n}{d})h(\frac{n}{d}))$, by $h(n) = h(d) + h(\frac{n}{d})$ for $d|n$.

LEMMA 7 One has $\sum_{d|n} \mu(d)v(d) = -\mu(n)$.

Proof: Apply Lemma 6 to $f(n) = 1$ for all n, $g = \mu$ and $h = v$. Then $(f * \mu).v = (f.v) * \mu + f * (\mu v)$, and by Lemma 1, $v(1) = 0$ and the commutativity of the (Dirichlet) convolution, we can write $(\mu v * f)(n) = \sum_{d|n} \mu(d)v(d) = -(v * \mu)(n) = -\mu(n)$ on base of Lemma 5.

We now extend the definition of function v to the rationals:

DEFINITION 2: Let $m, n \geq 1$ be integers and put $v(\frac{m}{n}) = v(m) - v(n)$ (The new function will be denoted by v, too.)

LEMMA 8: $v(n^t) = t.v(n)$ for all $n \geq 1$ and all integers t. Moreover, v is completely additive on the set of positive rational numbers.
Proof: For $t > 0$, this holds by Lemma 4. Let $t = -m$ $(m > 0)$. Then

$$v(n^t) = v(\frac{1}{n^m}) = v(1) - v(n^m) = 0 - mv(n) = tv(n).$$

Particularly, $v(n^{-1}) = -v(n)$. On the other hand,

$$v(\frac{m}{n}.\frac{p}{q}) = v(mp) - v(nq) = v(m) + v(p) - v(n) - v(q)$$

$$= (v(m) - v(n)) + (v(p) - v(q)) = v(\frac{m}{n}) + v(\frac{p}{q}).$$

Remark: P. Erdös has proved that when f is a completely additive and strictly increasing function, then $f(n) = c\log n$ (c - constant, see, e.g., [4], p. 204).

1. We now are able to prove the main result of this paper.
THEOREM: Let the functions u and v be defined as in Definitions 1 and 2. Assume that $\sum_{n\leq x} u(n) = O(x)$. Put $P_n = \sum_{k\in A(n)} \sin\frac{k\pi}{n}$. Then

$$\sum_{n\leq x} v(P_n) = -\frac{3v(2)}{\pi^2}.x^2 + O(x.\log x).$$

Proof: In Lemma 3 put

$$\psi(x) = \begin{cases} \sin\pi x, & \text{if } x \in (0,1) \\ 1, & \text{if } x = 1 \end{cases}$$

Then $F(n) = P_n \equiv \sum_{k\in A(n)} \sin\frac{k\pi}{n} > 0$ $(n \geq 2)$. It is well known that $\prod_{k=1}^{n-1} \sin\frac{k\pi}{n} = \frac{n}{2^{n-1}}$ (see, e.g., [10], p. 29). Therefore

$$P_n = \prod_{d|n} (\frac{\frac{n}{d}}{2^{\frac{n}{d}-1}}) = \frac{n^{\sum_{d|n}\mu(d)}}{\prod_{d|n} d^{\mu(d)}.2^{\sum_{d|n}\frac{n}{d}.\mu(d) - \sum_{d|n}\mu(d)}}.$$

Since $\sum_{d|n}\mu(d) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1 \end{cases}$, and the known formula $\varphi(n) = \sum_{d|n}\frac{\mu(d)}{d}$ (see, e.g., [6], p. 83) one obtains $P_n = \frac{1}{\prod_{d|n} d^{\mu(d)}.2^{\varphi(n)}}$, $(n \geq 2)$. By using Lemmas 7 and 8, we have

$$v(P_n) = -v(\prod_{d|n} d^{\mu(d)}.2^{\varphi(n)}) - \varphi(n)v(2) = \sum_{d|n}\mu(d)v(d) - \varphi(n)v(2) = u(n) - \varphi(n)v(2)$$

$(n \geq 2)$. The result follows by summation and Mertens' formula ([5], p. 268) $\sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2}.x^2 + O(x.\log x)$.

COROLLARY: 1. $$\lim_{m\to\infty} \frac{\sum_{n=1}^{m} v(P_n)}{m^2} = -\frac{3v(2)}{\pi^2}.$$

Remarks: 1. An example for the function v is given by $v(n) = \log n$ $(n \geq 1)$. Then $u(n) = \Lambda(n)$ - von Mangoldt's function. Sence it is known that $\sum_{n\leq x} \Lambda(n) = O(x)$ (see [1], p. 101), the above theorem gives

$$\sum_{n\leq x} \log P_n = -\frac{3\log 2}{\pi^2}.x^2 + O(x.\log x).$$

2. For the product $P_n = \prod_{k\in A(n)} \sin\frac{k\pi}{n}$ one has therefore

$$P_n = \begin{cases} \frac{p}{2^{\varphi(n)}}, & \text{if } n = p^a \\ \frac{1}{2^{\varphi(n)}}, & \text{otherwise} \end{cases} \quad \text{(see [7])}$$

3. By using the same procedure, we have $\prod_{k\in A(n)} \cos\frac{k\pi}{n} = (-\frac{1}{4})^{\frac{1}{2}\varphi(n)}$ if $n \geq 3$ is odd (see [2]).

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