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ON A EVEN PERFECT AND SUPERPERFECT NUMBERS

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1. A positive integer $n \geq 1$ is called perfect if $\sigma(n) = 2n$, where $\sigma(n)$ denotes the sum of divisors of n. Euclid proved that if n has the form $n = 2^k.(2^{k+1} - 1)$, with $k \geq 1$ and $2^{k+1} - 1$ prime, then n is a perfect. The reciprocal of this assertion was proved only 2000 years later by Euler. The classical monographs of number theory provide usually the proof given by Euler, or a variant of this, due to L.E.Dickson from 1913 (see [1,2]).

The aim of this Note is to obtain a new simple proof of Euler's theorem, by applying a method from [4]. A natural number $n \geq 1$ is called (after Suryanarajana) superperfect if holds that $\sigma(\sigma(n)) = 2n$. Suryanarajana [5] and Kanold [3] have proved that if n is an even superperfect number, then n has the form $n = 2^k$ $(k \geq 1)$, where $2^{k+1} - 1$ is prime. We will give in what follows a common proof for both results. This proof is based on the following

LEMMA For all integers $a, b \geq 1$ one has

$$\sigma(ab) \geq a\sigma(b) \tag{1}$$

with equality only for $a = 1$.

Proof: The proof of this assertion is very simple. Indeed, let $d|b$ be a divisor of b. Then $a.d$ is a divisor of ab, and writing this for each such divisor, after addition we get (1). For $a > 1$ we cannot have equality.

2. **The proof of Euler's theorem** The number n being even, put $n = 2^k.N$ with N odd. Equality $\sigma(n) = 2n$ can be written also as

$$(2^{k+1} - 1)\sigma(N) = 2^{k+1}.N. \tag{2}$$

Since $2^{k+1} - 1$ is an odd divisor of the right side, we must have

$$N = (2^{k+1} - 1).M \ (M \geq 1, \text{ integer}). \tag{3}$$

Now, from (2) one has $\sigma(N) = M.2^{k+1}$, which on base of (3) gives

$$M.2^{k+1} = \sigma(N) = \sigma((2^{k+1}.M) \geq M.\sigma(2^{k+1}.M) \geq M.2^{k+1}$$

with equality only for $M = 1$ and $2^{k+1} - 1$ - prime, since clearly $\sigma(s) \geq s + 1$ (with equality only for s - prime). Therefore, the assertion is proved.

3. <u>The proof of Suryanarajana's theorem</u> Let again $n = 2^k.N$ with N odd. One has $\sigma(n) = (2^{k+1} - 1)\sigma(N)$, where (1) implies

$$\sigma(\sigma(n)) = \sigma((2^{k+1} - 1)\sigma(N)) \geq \sigma(N)\sigma(2^{k+1}) \geq N.2^{k+1} = 2n$$

with equality only for $N = 1$ and $2^{k+1} - 1$ prime. The theorem is proved.

Remark One of the most difficult open problems of Number Theory is the problem of existence of an odd perfect or superperfect number.

References:

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[2] P. Erdös and J. Surányi, Selected chapters of number theory, Budapest, 1960 (Hungarian).

[3] H. J. Kanold, Über "Super-perfect numbers", Elem. Math. <u>24</u> (1969), 61-62.

[4] J. Sándor, On the composition of some arithmetic functions, Studia Univ. Babes-Bolyai, <u>34</u> (1989), 1, 7-14.

[5] D. Suryanarajana, Super-perfect numbers, Elem. Math. <u>24</u> (1969), 16-17.