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REMARKS ON φ, σ, ψ AND ρ FUNCTIONS

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φ and σ functions (see, e.g., [1]) are two of the most important arithmetic functions. They have a lot of very interesting properties. Some of them will be discussed below.

Firstly, following [2] we shall define for the natural number $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, where k, $\alpha_1, \alpha_2, \ldots \alpha_k \geq 1$ are natural numbers and $p_1, p_2, \ldots, p_k$ are different prime numbers, the foolowing functions:

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}.(p_i - 1),$$

$$\psi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}.(p_i + 1),$$

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i + 1} - 1}{p_i - 1},$$

$$\underline{set}(n) = \{p_1, p_2, \ldots, p_k\},$$

$$\underline{cas}(n) = k,$$

$$\underline{dim}(n) = \sum_{i=1}^{k} \alpha_i$$

Obviously, $\underline{dim}(n) \geq \underline{cas}(n)$ for every natural number n.

THEOREM 1 [3,4]: For every natural number $n \geq 5$:

$$\text{(a) } \sigma(n)^{\varphi(n)} < n^n < \varphi(n)^{\sigma(n)}$$

$$\text{(b) } \psi(n)^{\varphi(n)} < n^n < \varphi(n)^{\psi(n)}.$$

It can be seen easily, that the following inequalities

$$\psi(n)^{\varphi(n)} \leq \sigma(n)^{\varphi(n)} < n^n < \varphi(n)^{\psi(n)} \leq \varphi(n)^{\sigma(n)}$$

are valid, too. The following two inequalities follows directly:

$$\psi(n)^{\varphi(n)} < \varphi(n)^{\sigma(n)},$$

$$\sigma(n)^{\varphi(n)} < \varphi(n)^{\psi(n)}.$$

Therefore the inequalities

$$\varphi(n).\log(\psi(n)) < \sigma(n).\log(\varphi(n)),$$

$$\varphi(n).\log(\sigma(n)) < \psi(n).\log(\varphi(n)).$$

are hold, too. Hence, we obtain

$$(\psi(n).\sigma(n))^{\varphi(n)} < \varphi(n)^{\psi(n)+\sigma(n)}.$$

It can be easily seen, that function ψ is a modification of function φ. Function σ has an essentially different form that functions φ and ψ. Function σ can be written as:

$$\sigma(n) = \prod_{i=1}^{k} (p_i^{\alpha_i} + p_i^{\alpha_i - 1} + ... + p_i + 1).$$

Here we shall define another function, which is a modification of function σ. For the above defined natural number n, the new function has the form

$$\rho(n) = \prod_{i=1}^{k} (p_i^{\alpha_i} - p_i^{\alpha_i - 1} + ... + (-1)^{\alpha_i - 1} p_i + (-1)^{\alpha_i}).$$

Of course, for every natural number n :

$$\varphi(n) \leq \rho(n) < n < \psi(n) \leq \sigma(n).$$

Function ρ is multiplicative one.

THEOREM 2: For every natural number n

$$\text{a) } \varphi(n) + \sigma(n) \geq \psi(n) + \rho(n), \quad (5)$$

$$\text{b) } n + \varphi(n) \geq 2\rho(n), \quad (6)$$

$$\text{c) } \rho(n) + \psi(n) \geq 2n, \quad (7)$$

$$\text{d) } \varphi(n)^{\sigma(n)} \geq \rho(n)^{\psi(n)} > n^n > \psi(n)^{\rho(n)} \geq \sigma(n)^{\varphi(n)}. \quad (8)$$

Proof: a) Let $\underline{dim}(n) = 1$. Then (5) is obviously valid.

Let us assume that the assertion is valid for every natural numbers n and k, such that $\underline{dim}(n) \leq k$. Let $\overline{n} = n.p$, where $\underline{dim}(n) = k$ and p is a prime number. For p there are two cases again.

Case 1: $p \notin \underline{set}(n)$. Then,

$$\varphi(\overline{n}) + \sigma(\overline{n}) - \psi(\overline{n}) - \rho(\overline{n})$$

$$= \varphi(n).(p-1) + \sigma(n).(p+1) - \psi(n).(p+1) - \rho(n).(p-1)$$

$$= p.(\varphi(n) + \sigma(n) - \psi(n) - \rho(n)) + \sigma(n) - \psi(n) + \rho(n) - \varphi(n) \geq 0.$$

Case 2: $p \in \underline{set}(n)$. Then $n = m.p^a$ for some natural numbers $m, p \geq 1$ and $\overline{n} = m.p^{a+1}$.

$$\varphi(\overline{n}) + \sigma(\overline{n}) - \psi(\overline{n}) - \rho(\overline{n})$$

$$= \varphi(n).p + \sigma(n).\frac{p^{a+2}-1}{p^{a+1}-1} - \psi(n).p - \rho(n).\frac{p^{a+1} - p^a + ... + (-1)^a p + (-1)^{a+1}}{p^a - p^{a-1} + ... + (-1)^{a-1}p + (-1)^a}$$

$$= p.(\varphi(n)+\sigma(n)-\psi(n)-\rho(n))+\sigma(n).\frac{p-1}{p^{a+1}-1} - \rho(n).\frac{(-1)^{a+1}}{p^a - p^{a-1} + ... + (-1)^{a-1}p + (-1)^a}$$

$$= \sigma(m).\frac{p-1}{p^{a+1}-1}.\frac{p^{a+1}-1}{p-1}$$

$$-\rho(m).(p^a - p^{a-1} + ... + (-1)^{a-1}p + (-1)^a).\frac{(-1)^{a+1}}{p^a - p^{a-1} + ... + (-1)^{a-1}p + (-1)^a}$$

$$= \sigma(m) - \rho(m).(-1)^{a+1} \geq \sigma(m) - \rho(m) > 0.$$

b) Let $\underline{dim}(n) = 1$. Then (6) is obviously valid.

Let us assume that the assertion is valid for every natural numbers n and k, such that $\underline{dim}(n) \leq k$. Let $\overline{n} = n.p$, where $\underline{dim}(n) = k$ and p is a prime number.

Case 1: $p \notin \underline{set}(n)$. Then,

$$\overline{n} + \varphi(\overline{n}) - 2\rho(\overline{n}) = np + \varphi(n)(p-1) - 2\rho(n)(p-1) > 0.$$

Case 2: $p \in \underline{set}(n)$. Then $n = m.p^a$ for some natural numbers $m, p \geq 1$ and $\overline{n} = m.p^{a+1}$.

$$\overline{n} + \varphi(\overline{n}) - 2\rho(\overline{n}) = m.p^{a+1} + \varphi(n).p^a.(p-1) - 2\rho(m).(p^{a+1} - p^a + ... + (-1)^{a+1})$$

$$= p^{a+1}.(m + \varphi(m) - 2\rho(m)) - \varphi(m).p^a + 2\rho(m).(p^a + ... + (-1)^a)$$

$$> p^{a-1}.(2\rho(n).(p-1) - \varphi(n).p) > 0,$$

because $2(p-1) \geq p$ and $\rho(n) \geq \varphi(n)$.

c) and d) are proved analogically.

References:

[1] Chandrasekharan K. Introduction to Analytic Number Theory. Springer-Verlag, Berlin, 1968.

[2] Atanassov K., Short proof of a hypothesis of A. Mullin. *Bull. of Number Theory and Related Topics,* Vol. IX (1985), No. 2, 9-11.

[3] Atanassov K. New integer functions, related to φ and σ functions. *Bull. of Number Theory and Related Topics,* Vol. XI (1987), No. 1, 3-26.

[4] Aranassov K. Remarks on φ, σ and ψ functions. *Mathematical Forum* (in press, 2001).