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Expressions for the Dirichlet Inverse of an Arithmetical Function

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Abstract

We express the values of the Dirichlet inverse f^{-1} in terms of the values of f without using the values of f^{-1}. We use a method based on representing $f^{-1} * f = \delta$ as a system of linear equations. Jagannathan has given many of the results of this paper without proof starting from the basic recurrence relation for the values of f^{-1}.

1 Introduction

By an arithmetical function we mean a complex-valued function of a real variable, which is zero if the argument is not a positive integer. The Dirichlet convolution of two arithmetical functions f and g is defined as

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d). \tag{1.1}$$

The Dirichlet convolution is associative and commutative. The function δ, defined as $\delta(1) = 1$ and $\delta(n) = 0$ for $n > 1$, serves as the identity under the Dirichlet convolution. The Dirichlet inverse of an arithmetical function f is the arithmetical function f^{-1} such that

$$f * f^{-1} = f^{-1} * f = \delta. \tag{1.2}$$

An arithmetical f function possesses the Dirichlet inverse if and only if $f(1) \neq 0$. From (1.1) and (1.2) we obtain the usual recursive expression for the Dirichlet inverse as $f^{-1}(1) = 1/f(1)$ and for $n > 1$

$$f^{-1}(n) = \frac{-1}{f(1)} \sum_{\substack{d|n \\ d<n}} f^{-1}(d)f(n/d). \tag{1.3}$$

In this paper we present some explicit expressions for the function values $f^{-1}(n)$. To be more precise, we express $f^{-1}(n)$ in terms of the values of f without using the values of f^{-1}.

We adopt a known method (see e.g. [4], [5], [6], [8]) based on representing (1.2) as a system of linear equations. As an example we give expressions for the Möbius function which is the Dirichlet inverse of the constant function 1. Jagannathan [3] has given many of the results of this paper without proof starting from the basic recurrence relation for the values of f^{-1}.

2 Expressions

Let f be an arithmetical function possessing the Dirichlet inverse, that is, with $f(1) \neq 0$. It is easy to see that $(af)^{-1} = \frac{1}{a}f^{-1}$ for any nonzero complex constant a. Therefore, in deriving expressions for $f^{-1}(n)$, we may without loss of generality assume that $f(1) = 1$.

Theorem 2.1 *Let f be an arithmetical function with $f(1) = 1$. For $n \geq 2$,*

$$f^{-1}(n) = (-1)^{n-1} \det \begin{pmatrix} f(2) & 1 & 0 & 0 & \cdots & 0 \\ f(3) & 0 & 1 & 0 & \cdots & 0 \\ f(4) & f(2) & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ f(n-1) & f(\frac{n-1}{2}) & f(\frac{n-1}{3}) & f(\frac{n-1}{4}) & \cdots & 1 \\ f(n) & f(\frac{n}{2}) & f(\frac{n}{3}) & f(\frac{n}{4}) & \cdots & f(\frac{n}{n-1}) \end{pmatrix}_{(n-1)\times(n-1)} \tag{2.1}$$

or

$$f^{-1}(n) = (-1)^{n-1} \det(f(\frac{i+1}{j}))_{(n-1)\times(n-1)}.$$

Proof Applying (1.2) at $1, 2, \ldots, n$ and noting that $f(1) = 1$ we obtain a system of n linear equations with n unknowns $f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(n)$ as

$$\begin{cases} 1 \cdot f^{-1}(1) = 1, \\ f(2)f^{-1}(1) + 1 \cdot f^{-1}(2) = 0, \\ f(3)f^{-1}(1) + 0 \cdot f^{-1}(2) + 1 \cdot f^{-1}(3) = 0, \\ f(4)f^{-1}(1) + f(2)f^{-1}(2) + 0 \cdot f^{-1}(3) + 1 \cdot f^{-1}(4) = 0 \\ \qquad \cdots \end{cases} \tag{2.2}$$

This can be written as

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ f(2) & 1 & 0 & 0 & \cdots \\ f(3) & 0 & 1 & 0 & \cdots \\ f(4) & f(2) & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{n\times n} \begin{pmatrix} f^{-1}(1) \\ f^{-1}(2) \\ f^{-1}(3) \\ f^{-1}(4) \\ \vdots \end{pmatrix}_{n\times 1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}_{n\times 1}.$$

By Cramér's rule,

$$f^{-1}(n) = \det\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 1 \\ f(2) & 1 & 0 & 0 & \cdots & 0 \\ f(3) & 0 & 1 & 0 & \cdots & 0 \\ f(4) & f(2) & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}_{n\times n}.$$

Expanding by the nth column we obtain (2.1). □

Theorem 2.2 *Let f be an arithmetical function with $f(1) = 1$. For $n \geq 2$,*

$$f^{-1}(n) = \sum_{k=1}^{\Omega(n)} (-1)^k \sum_{\substack{d_1 d_2 \cdots d_k = n \\ d_1, d_2, \ldots, d_k > 1}} f(d_1) f(d_2) \cdots f(d_k), \qquad (2.3)$$

where $\Omega(n)$ is the number of prime divisors of n, each counted according to its multiplicity.

Proof Expression (2.3) follows by expanding the determinant expression (2.1). For each $k = 1, 2, \ldots, \Omega(n)$ in the sum of (2.3), $n - 1 - k$ indicates the number of 1's of the upper diagonal taken to the product terms of the expansion of the determinant in (2.1). The details of the verification are as follows.

If we take all the $n - 2$ 1's of the upper diagonal, we must also take the element $f(n)$ and we arrive at the term $(-1)^{n-2} 1^{n-2} f(n)$ or $(-1)^{n-2} f(n)$, where $(-1)^{n-2}$ is the parity of the respective permutation. This term in the determinant expansion corresponds to the index $k = 1$ in the sum (2.3). If we take the 1's of the upper diagonal except for the ith 1, then we must also take the elements $f(i+1)$ and $f(\frac{n}{i+1})$ and we thus arrive at the term $(-1)^{n-3} 1^{n-3} f(i + 1) f(\frac{n}{i+1})$, where $(-1)^{n-3}$ is the parity of the respective permutation. Letting i run through the integers $1, 2, \ldots, n-2$ we obtain the term

$$(-1)^{n-3} \sum_{i=1}^{n-2} f(i+1) f(\frac{n}{i+1})$$

or

$$(-1)^{n-3} \sum_{\substack{d_1 d_2 = n \\ d_1, d_2 > 1}} f(d_1) f(d_2).$$

This term in the determinant expansion corresponds to the index $k = 2$ in the sum (2.3). Proceeding in this way we see that the determinant in (2.1) is equal to the sum in (2.3) multiplied by $(-1)^{n-1}$. □

Theorem 2.3 *Let f be an arithmetical function with $f(1) = 1$. Let n be an arbitrary but fixed positive integer (≥ 2). Let $e_1, e_2, \ldots, e_m$ be the divisors of n, which are > 1. Then*

$$f^{-1}(n) = \sum_{\substack{l_1, l_2, \ldots, l_m \geq 0 \\ e_1^{l_1} e_2^{l_2} \cdots e_m^{l_m} = n}} (-1)^{\sum_{i=1}^{m} l_i} \frac{\left(\sum_{i=1}^{m} l_i\right)!}{\prod_{i=1}^{m} (l_i!)} \prod_{i=1}^{m} f(e_i)^{l_i}. \qquad (2.4)$$

Proof This is a rearrangement of the sum in (2.3). In (2.3) the sum is over ordered sets of divisors of n, while in (2.4) the sum is over unordered sets of divisors of n. The factor $(\sum_{i=1}^m l_i)!/\prod_{i=1}^m (l_i!)$ counts the number of ways to order the corresponding divisors. □

We could also apply (1.2) at points other than $1, 2, \ldots, n$. For example, assume that f is a multiplicative function. Then the inverse f^{-1} is completely determined by its values at prime powers. Thus, applying (1.2) at $1, p, \ldots, p^n$ we obtain the expressions

$$f^{-1}(p^n) = (-1)^n \det \begin{pmatrix} f(p) & 1 & 0 & \cdots & 0 \\ f(p^2) & f(p) & 1 & \therefore & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f(p^{n-1}) & f(p^{n-2}) & f(p^{n-3}) & \cdots & 1 \\ f(p^n) & f(p^{n-1}) & f(p^{n-2}) & \cdots & f(p) \end{pmatrix}_{n\times n}, \tag{2.5}$$

$$f^{-1}(p^n) = \sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1+i_2+\cdots+i_k=n \\ i_1, i_2, \ldots, i_k > 0}} f(p^{i_1}) f(p^{i_2}) \cdots f(p^{i_k}), \tag{2.6}$$

$$f^{-1}(p^n) = \sum_{\substack{l_1, l_2, \ldots, l_n \geq 0 \\ l_1+2l_2+\cdots+nl_n=n}} (-1)^{\sum_{i=1}^{n} l_i} \frac{\left(\sum_{i=1}^{n} l_i\right)!}{\prod_{i=1}^{n} (l_i)!} \prod_{i=1}^{n} f(p^i)^{l_i}. \tag{2.7}$$

3 Expressions for the Möbius function

The Möbius function μ is the Dirichlet inverse of the constant function 1. The classical expression for the Möbius function is

$$\mu(n) = \begin{cases} (-1)^n & \text{if } n = p_1 p_2 \cdots p_n,\ p_i \neq p_j\ (i \neq j), \\ 0 & \text{if there exists a prime } p \text{ such that } p^2 | n. \end{cases}$$

Theorem 3.1 *For $n \geq 2$,*

$$\mu(n) = (-1)^{n-1} \det \left(\zeta\left(\frac{i+1}{j}\right) \right)_{(n-1)\times(n-1)},$$

where $\zeta(x) = 1$ if x is a positive integer and $\zeta(x) = 0$ otherwise.

Proof Theorem 3.1 is a direct consequence of Theorem 2.1. □

Definition 3.1 For $k \geq 1$, let $\Delta_k(n)$ denote the number of k-tuples $(d_1, d_2, \ldots, d_k)$ such that $d_1 d_2 \cdots d_k = n$, $d_1, d_2, \ldots, d_k > 1$. In addition, let $\Delta_0 = \delta$, that is, $\Delta_0(1) = 1$ and $\Delta_0(n) = 0$ for $n > 1$.

Theorem 3.2 *For $n \geq 1$,*

$$\mu(n) = \sum_{k=0}^{\Omega(n)} (-1)^k \Delta_k(n). \tag{3.1}$$

Proof Theorem 3.2 is a direct consequence of Theorem 2.2. □

Theorem 3.2 prompts us to study the function $\Delta_k(n)$ in more detail. We shall express $\Delta_k(n)$ in terms of the well-known function $\tau_k(n)$ (see [2], [7, Chapter IV]). For $k \geq 1$, $\tau_k(n)$ is defined as the number of k-tuples $(d_1, d_2, \ldots, d_k)$ such that $d_1 d_2 \cdots d_k = n$. In other words, for $k \geq 1$, $\tau_k = \zeta * \zeta * \cdots * \zeta$ (ζ, k times). In addition, it is convenient to define $\tau_0 = \delta$. It should be noted that some authors (see e.g. [7]) use the notation $d_k(n)$ for $\tau_k(n)$. Also note that $\tau_2(n)$ is the classical divisor function, usually denoted by $\tau(n)$ or $d(n)$.

Theorem 3.3 *For $k \geq 0$*

$$\Delta_k(n) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \tau_{k-i}(n), \quad n \geq 1. \tag{3.2}$$

Proof If $k = 0$, both sides of (3.2) reduce to $\delta(n)$. Let $k \geq 1$. Denote

$$\begin{aligned} S_k(n) &= \{(d_1, d_2, \ldots, d_k) \mid d_1 d_2 \cdots d_k = n\} \\ S_{k,j}(n) &= \{(d_1, d_2, \ldots, d_k) \mid d_1 d_2 \cdots d_k = n, d_j > 1\}, \quad j = 1, 2, \ldots, k. \end{aligned}$$

Then

$$\Delta_k(n) = \operatorname{card}\left(\bigcap_{j=1}^{k} S_{k,j}(n)\right).$$

By the inclusion-exclusion principle

$$\Delta_k(n) = \operatorname{card}(S_k(n)) + \sum_{i=1}^{k} (-1)^i \sum_{1 \leq j_1 < \cdots < j_i \leq k} \operatorname{card}\left(\overline{S_{k,j_1}(n)} \cap \cdots \cap \overline{S_{k,j_i}(n)}\right),$$

where $\overline{S_{k,j}(n)}$ is the complement of $S_{k,j}(n)$ in $S_k(n)$. We thus obtain

$$\begin{aligned} \Delta_k(n) &= \tau_k(n) + \sum_{i=1}^{k} (-1)^i \binom{k}{i} \tau_{k-i}(n) \\ &= \sum_{i=0}^{k} (-1)^i \binom{k}{i} \tau_{k-i}(n). \end{aligned}$$

This completes the proof of Theorem 3.3. □

Corollary For $k \geq 0$

$$\tau_k(n) = \sum_{i=0}^{k} \binom{k}{i} \Delta_i(n), \quad n \geq 1. \tag{3.3}$$

Proof Application of the classical binomial inversion formula (see [1, p. 96]) to (3.2) gives (3.3). □

Theorem 3.4 *We have*

$$\mu(n) = \sum_{k=0}^{\Omega(n)} (-1)^k \sum_{i=0}^{k} (-1)^i \binom{k}{i} \tau_{k-i}(n), \quad n \geq 1. \tag{3.4}$$

If the canonical representation of n *is* $n = \prod_{j=1}^{r} p_j^{\alpha_j}$, *then*

$$\mu(n) = \sum_{k=1}^{\Omega(n)} (-1)^k \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} \prod_{j=1}^{r} \binom{\alpha_j + k - i - 1}{\alpha_j}, \quad n \geq 2, \tag{3.5}$$

where $\Omega(n) = \alpha_1 + \alpha_2 + \cdots + \alpha_r$.

Proof Equation (3.4) follows from (3.1) and (3.2). Equation (3.5) follows from (3.4) and the property $\tau_k(p^\alpha) = \binom{\alpha+k-1}{\alpha}$. □

Remark From (2.5) and (2.6) we obtain the expressions

$$\begin{aligned}
\mu(p^n) &= (-1)^n \det \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \\
&= \sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1+i_2+\cdots+i_k=n \\ i_1,i_2,\ldots,i_k>0}} 1 \\
&= \sum_{k=1}^{n} (-1)^k \binom{n-1}{k-1} \\
&= \begin{cases} -1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}
\end{aligned}$$

Remark Jagannathan [3] has given Equations (2.3), (2.4), (3.1), (3.3) and (3.5) without proof.

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