NNTDM 6 (2000) 4, 113-117

ANALYSIS OF THE ROOTS OF SOME CARDANO CUBICS

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ABSTRACT

The Cardano cubic, $y^3 - 6pqy - 3pq(p+q)$, $p, q \in \mathbf{Z}_+$, has one real zero and a complex conjugate pair. The real zero is given by $2(2pq+e)^{\frac{1}{2}}$ or $(E+2)(2pq)^{\frac{1}{2}}$, in which e, E are important parameters that feature in the roots of all Cardano cubics. They are functions of the coefficients of the complex conjugate pairs. We find that

$$\begin{aligned} e &= \frac{2q^2 R \tan^2\theta}{3 - \tan^2\theta}, \\ E &= 2\left\{\left(\frac{3}{3-\tan^2\theta}\right)^{\frac{1}{2}} - 1\right\} \end{aligned}$$

with $R = \frac{p}{q} = h(\theta)$ and $11^o < \theta < 60^o$ for real zero. Futhermore, for E integer, the range of θ is reduced to $52^o < \theta < 60^o$, where the functional surfaces suggest the reason the integer E would only be compatible with an irrational value of R. This is verified algebraically.

1 INTRODUCTION

We have recently presented the Cardano family of Equations [2] in which the member of lowest degree, the cubic is given by the canonical form

$$y^3 - 6pqy - 3pq(p+q) = 0, \tag{1.1}$$

$p, q \in \mathbf{Z}_+$. For an extensive study of the coefficients of the Cardano cubic, when all roots are real, the interested reader is directed to [1].

The higher family members are consistent with the cubic in that they have only one real root (n odd) or two (n even) and complex conjugate pairs. The real roots have the forms $(n-1)(2pq+e)^{\frac{1}{2}}$ and, additionally, $-(2pq+d)^{\frac{1}{2}}$ when n is even. The positive root may also have the form $(E+n-1)(2pq)^{\frac{1}{2}}$. The parameters e and E are always non-integer when $p, q \in \mathbf{Z}_+$ [2] and are functions of the complex conjugate pair.

The purpose of the present paper is to analyse e and E in the cubic case, principally in relation to the trigonometric functions associated with the complex conjugate pair.

2 THE PARAMETER e

As shown previously [2], from the theory of equations and with the complex conjugate pair of Equation (1.1) given by $(a \pm bi)$,

$$e = \frac{b^2}{3} \tag{2.1}$$

and

$$e = a^2 - 2pq, \tag{2.2}$$

so that

$$a^2 + b^2 = 4e + 2pq \tag{2.3}$$

or

$$a^2 + b^2 = 4a^2 - 6pq. \tag{2.4}$$

As well, from the theory of complex numbers,

$$a = r\cos\theta \tag{2.5}$$

$$b = r\sin\theta \tag{2.6}$$

which yield

$$r^2 = a^2 + b^2$$

From Equation (2.4)

$$r^2 = 4r^2\cos^2\theta - 6pq \tag{2.7}$$

so that

$$r^2 = \frac{6pq}{4\cos^2\theta - 1}. \tag{2.8}$$

From Equations (2.3) and (2.8),

$$e = \frac{2pq(1-\cos^2\theta)}{4\cos^2\theta - 1} = \frac{2pq}{3\cot^2\theta - 1} \tag{2.9}$$

and so on. Since p and q can have multiple values for constant pq, we use $R = \frac{p}{q}$, with $p < q$, $0 < R < 1$. Thus

$$e = \frac{2q^2 R\tan^2\theta}{3 - \tan^2\theta} \tag{2.10}$$

Obviously, there are constraints on θ since the denominator in Equation (2.8) is zero for $\theta = \frac{n\pi}{3}$, $n \in \mathbf{Z}$, $n \not| 3$. It follows that

$$R = \frac{48 - 4Q}{f(\theta)} - 1 \tag{2.11}$$

with

$$Q = \{6(24 - f(\theta))\}^{\frac{1}{2}}$$

and

$$f(\theta) = \cos^4\theta(3 - \tan^2\theta)^3.$$

Thus e is now a function of q and θ only. Since $f(\theta) < 24$ for a real non-zero solution of Q, $\theta > 11.266345^o$ is another constraint on θ.

Finally, we can express

$$e = q^2 F(\theta) \tag{2.12}$$

with

$$F(\theta) = \frac{8\tan^2\theta(12 - Q)}{f(\theta)(3 - \tan^2\theta)} - \frac{2\tan^2\theta}{3 - \tan^2\theta}. \tag{2.13}$$

Since q can have any integer value, $F(\theta)$ must always be irrational for $p, q \in \mathbf{Z}_+$, since the rational root of equation (1.1) is always non-integer [2].

If we differentiate $F(\theta)$ with respect to θ in Equation (2.13), then we find that

$$\frac{dF}{d\theta} = F(\theta)\left\{\frac{1}{R}\frac{dR}{d\theta} + \frac{6\sec^2\theta}{\tan\theta(3 - \tan^2\theta)}\right\} \tag{2.14}$$

with

$$\frac{1}{R}\frac{dR}{d\theta} = \frac{-24\cos\theta\sin\theta(8\cos^2\theta + 1)}{\cos^4\theta(3 - \tan^2\theta)Q}. \tag{2.15}$$

At stationary point of F

$$\frac{1}{R}\frac{dR}{d\theta} = \frac{-6\sec^2\theta}{\tan\theta(3 - \tan^2\theta)}, \tag{2.16}$$

so that Equation (2.14) reduces to the following quartic in $\cos^2\theta$:

$$704\cos^8\theta - 1040\cos^6\theta + 228\cos^4\theta + 109\cos^2\theta + 8 = 0. \tag{2.17}$$

This gives values of θ as 12.059493^o, 28.791636^o. As well, a stationary point might be expected to occur when $F(\theta) = 0$, that is for $\theta = 60^o$, but this is outside the range because the denominator in Equation (2.13) is then zero.

e may be expressed as the reduced cubic [2]:

$$s^3 - \left(\frac{3}{4}(pq)^2\right)s + \left(\frac{1}{4}(pq)^3 - \frac{9}{64}(pq)^2(p+q)^2\right) = 0 \tag{2.18}$$

with $s = e + pq$. Substituting $e = \dfrac{2pq}{3\cot^2\theta - 1}$ from Equation (2.9), and simplifying with $t = 1 + 2\cos 2\theta$, we obtain

$$t^3 + \mathbf{P}t + \mathbf{Q} = 0 \tag{2.19}$$

where $\mathbf{P} = \mathbf{Q} = \dfrac{-24pq}{(p+q)^2} = \dfrac{-24R}{(R+1)^2}$. This yields θ as a simpler function of p, q. If $27Q^2 + 4P^3 < 0$, Equation (2.19) has three real roots; otherwise it has one real root and a complex conjugate pair. Since $\mathbf{P} = \mathbf{Q}$, the parity of $27(p^2+q^2) - 42pq$ will indicate the types of roots. Since

$$27(p^2+q^2) > 21(p^2+q^2) > 42pq,$$

there will only be one real root for t. When $R = 1$, we find $\theta = 11.266345^o$ in Equation (2.19), and when $R = 0$, $\theta = 60^o$ as before.

3 THE PARAMETER E

The trigonometric function for E is

$$E = 2\left\{\left(\frac{3}{3-\tan^2\theta}\right)^{\frac{1}{2}} - 1\right\} \tag{3.1}$$

in which the constraints for θ are the same as for e, namely $11.266345^o < \theta < 60^o$. Since $E = 1$ when $\theta = 52.238756^o$, there is only a relatively small range for θ where E is an integer, namely, $52^o < \theta < 60^o$. However, even within this range, since $y_0 \notin \mathbf{Z}$ when $p, q \in \mathbf{Z}_+$, $(2pq)^{\frac{1}{2}} \notin \mathbf{Z}$, when $E \in \mathbf{Z}$. The curve for θ versus $\ln E$ resembles a distorted $\tanh x$ curve, with $x = \ln E$, that is,

$$\tanh x = 1 - \frac{2}{\exp(2\ln E) + 1} \tag{3.2}$$

with $\theta = 11^o$ and 60^o as its asymptotes.

For the range of interest of θ, namely, $52^o - 60^o$, $\tanh x$ is linear in θ so that θ can be replaced by the hyperbolic function in this range. It is also of interest to note that $R = 0$ at $\theta = 60^o$ and $R = 1$ at $\theta = 11.266345^o$.

Another feature of E is that the surface defined by (θ, R, E) is elliptical for $\theta > 50^o$. One would expect that all the integer points for E on this surface would be associated with irrational values of R. Using Equations (2.11) and (3.1) we get

$$R + 1 = \frac{4}{9}g(g^2-3)\left\{g(g^2-3) - \left[g^2(g^2-3)^2 - \frac{9}{2}\right]^{\frac{1}{2}}\right\} \tag{3.3}$$

where $g = E + 2$. Table 1 lists values of E and the corresponding θ and R values. By $E = 10$, θ has already reached 59.65^o. Thus, between 59.65^o and the limit 60^o an infinity of integer values for E can be fitted in, since even at $E = 1000$ the angle differences with 60^o are extremely small.

Equation (3.3) has two opposing components with $g(g^2-3)$ tending to infinity as the other approaches zero. This increases errors in the estimate. However, Equation (3.3) may be expressed as

$$9\left(R + \frac{1}{R}\right) = 8\left(g(g^2-3)\right)^2 - 18, \tag{3.4}$$

Table 1

E	1	2	3	4	5	6	7
θ	52.238756	56.309932	57.791501	58.517846	58.932547	59.193019	59.367827
R^*	0.0034967	0.0004171	0.0001040	0.0000443	0.0000776	0.0000443	0.0001784
R^{**}	0.0034967	0.0004164	0.0000937	0.0000287	0.0000109	0.0000047	0.0000023
E	8	9	10	100	1000	5000	
θ	59.491041	59.581247	59.649318	59.995230	59.999951	59.999998	
R^*	0.0000743	0.0012481	0.0026416	~ 0	~ 0	~ 0	
R^{**}	0.0000012	0.0000007	0.0000004	~ 0	~ 0	~ 0	

$*$ R values from Equation (3.3),
$**$ R values from Equation (3.4).

so that if $g \in \mathbf{Z}$, then $(R + \frac{1}{R}) \in \mathbf{Z}$ or have a denominator equal to 3 or 9. If $R \in \mathbf{Q}$ with $R = \dfrac{n}{m}$, $n \neq m$, then

$$R + \frac{1}{R} = \frac{(n+m)^2}{mn} - 2. \tag{3.5}$$

Since $n \neq m$, the right hand side of Equation (3.5) can never be an integer in this case. Thus R cannot be a rational fraction and is irrational whenever E is an integer. If R has 3 or 9 as denominator, then $g(g^2 - 3)$ will not be an integer.

For alternative approaches to the treatment of Cardano's solution of the cubic, the reader is referred to the classic exposition of Turnbull [3].

Gratitude is expressed to Dr C.K. Wong of University of Technology, Sydney, and Prince of Wales Hospital, Randwick, for technical assistance and advice.

References

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[3] Turnbull, H.W. 1957. *Theory of Equations.* Fifth Edition. Edinburgh: Oliver and Boyd.

AMS Classification Numbers: 11C08, 11D41.