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AN ELEMENTARY IDENTITY

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Let us have three arbitrary positive real numbers a, b and c for which holds

$$abc = 1. \tag{1}$$

Here we shall prove the validity of the following identity

$$(a - 1 + \frac{1}{b}).(b - 1 + \frac{1}{c}).(c - 1 + \frac{1}{a}) = (a + 1 - \frac{1}{b}).(b + 1 - \frac{1}{c}).(c + 1 - \frac{1}{a}). \tag{2}$$

It is possible, the shortest proof is the following. Using (1) we can homogenize (2), taking the arbitrary positive real numbers x, y and z and putting

$$a = \frac{x}{y},\ b = \frac{y}{z},\ c = \frac{z}{x}.$$

Then

$$a.b.c = \frac{x}{y}.\frac{y}{z}.\frac{z}{x} = 1,$$

i.e., (1) is valid and

$$(a - 1 + \frac{1}{b}).(b - 1 + \frac{1}{c}).(c - 1 + \frac{1}{a}) = (\frac{x}{y} - 1 + \frac{z}{y}).(\frac{y}{z} - 1 + \frac{x}{z}).(\frac{z}{x} - 1 + \frac{y}{x})$$

$$= \frac{1}{xyz}.(x - y + z).(y - z + x).(z - x + y) = \frac{1}{xyz}.(x + y - z).(y + z - x).(z + x - y)$$

$$= (\frac{x}{y} + 1 - \frac{z}{y}).(\frac{y}{z} + 1 - \frac{x}{z}).(\frac{z}{x} + 1 - \frac{y}{x}) = (a + 1 - \frac{1}{b}).(b + 1 - \frac{1}{c}).(c + 1 - \frac{1}{a})$$

with which (2) is proved.

The opposite assertion *if (2) is valid for arbitrary positive real numbers a, b and c, then (1) is not valid* does not hold. Because, for example, if

$$a = \frac{4}{3},\ b = \frac{1}{3},\ c = \frac{1}{3}$$

then

$$(a - 1 + \frac{1}{b}).(b - 1 + \frac{1}{c}).(c - 1 + \frac{1}{a}) = \frac{70}{108} = (a + 1 - \frac{1}{b}).(b + 1 - \frac{1}{c}).(c + 1 - \frac{1}{a}),$$

but

$$a.b.c = \frac{4}{27} \neq 1.$$