NNTDM 6 (2000), 88-99

Enumeration of integer k-gons with perimeter n

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October 12, 2000

Abstract

We consider the following enumeration problem : how many are there "distinct" k-gons with integer sides and perimeter n. A solution is known for $k = 3$ when "distinct" means "non-congruent" . This does not hold in the general case since for $k > 3$ a k-gon is not uniquely determined by its side lengths. We define the concept "distinct" in an appropriate way and reduce the problem to an enumeration of all distinct integer labels on the sides of a fixed regular k-gone satisfying a given condition. This enumeration can be done by the well known Polya's Theory of Counting. The simple structure of the considered objects (a regular k-gon and the dihedral group of order k) allows us to prove our results in an alternative way using only elementary concepts and techniques from Group Theory and Number Theory.

1 Introduction

In this paper we consider the following enumeration problem :

Find $T(k,n)$, the number of all "distinct" k – gons, having integer sides and perimeter n.

The simplest case, $k = 3$, is solved in [5]. An integer triangle with perimeter n is uniquely determined by the triple $(a, b, c) \in Z_+^3$ of its side lengths (Z_+ is the set of the positive integers). Such a triple must satisfy

$$a < b + c,\ b < a + c,\ c < a + b,\ a + b + c = n$$

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and the triples (a, c, b), (b, a, c), (b, c, a), (c, a, b), and (c, b, a) are representations of the same triangle. Therefore we obtain that

$$T(3, n) = \left|\left\{(a, b, c) \mid (a, b, c) \in Z_+^3, \ a \geq b \geq c, \ a < n - a\right\}\right|$$

which is an elementary enumeration problem.

The general case, i.e., for arbitrary $k \geq 3$ is not so easily solved. Since a k-gon is not uniquely determined by its side lengths for $k > 3$ we cannot replace "distinct" with "non-congruent" as in the case $k = 3$. That is why we need a reasonable definition of this concept. We shall assume that two k-gons $\mathcal{P} = P_1P_2 \ldots P_k$ and $\mathcal{Q} = Q_1Q_2 \ldots Q_k$ are **non-distinct** iff 1, 2 or 3 holds:

1. There exist a congruence that maps $\mathcal{P}$ onto $\mathcal{Q}$.
2. $|P_jP_{j+1}| = |Q_jQ_{j+1}|$ for every $j = 1, 2, \ldots k$ (setting $P_{k+1} = P_1$ and $Q_{k+1} = Q_1$).
3. There exists a k-gon $\mathcal{R}$ such that $\mathcal{P}$ and $\mathcal{R}$ are non-distinct by 1 and $\mathcal{R}$ and $\mathcal{Q}$ are non-distinct by 2

Of course, we shall assume that two k-gons are **distinct** iff they fail to be non-distinct.

This definition of the concept "distinct" allows us to obtain a bijection between all distinct k-gons and all **distinct labels** $x = (x_1, x_2, \ldots x_k)$ on the sides of a fixed regular k-gon satisfying the condition

$$0 < x_j < \sum_{1 \leq i \leq k, \, i \neq j} x_i = \sum_{i=1}^{k} x_i - x_j \text{ for } j = 1, 2, \ldots k$$

(assuming that two labels x' and x'' are distinct if there exists no congruence Φ mapping the regular k-gon onto itself and such that $\Phi\left(x'\right) = x''$). The bijection is defined as

$$P_1P_2 \ldots P_{k-1}P_k \longleftrightarrow \left(|P_1P_2|, \, |P_2P_3|, \ldots |P_{k-1}P_k|, \, |P_kP_1|\right)$$

(in Figure 1 the pentagons $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ are non-distinct - $\mathcal{P}$ and $\mathcal{R}$ by 1, $\mathcal{P}$ and $\mathcal{Q}$ by 2, $\mathcal{Q}$ and $\mathcal{R}$ by 3. The regular labelled pentagon $\mathcal{A}$ is bijective to all of them).

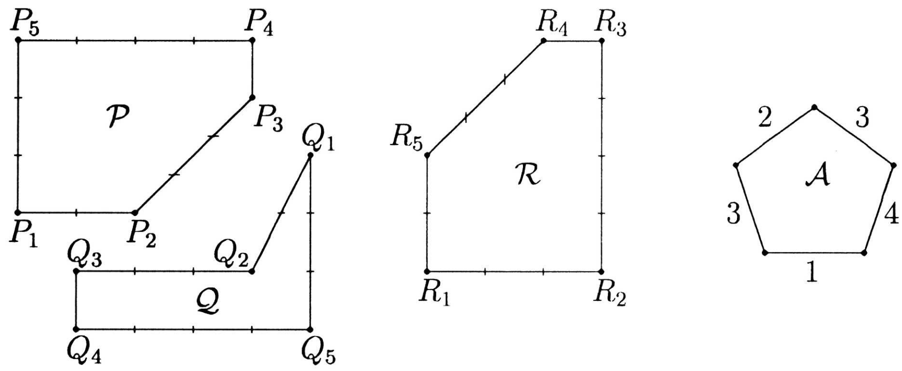

Figure 1

Thus we reduce our problem to the following :

Find $T(k,n)$, the number of all distinct integer labels
on the sides of the regular k − gon, satisfying the conditions :
$1 \leq x_j < n - x_j$ for $j = 1, 2, \ldots k$ and $\sum_{j=1}^{m} x_j = n$.

The latter problem can be solved directly by the well known Pólya's Theory of Counting - [6], [1]. There exists a standard way to prove the Pólya's theorem or some of its generalization and it is presented in [2]. We preferred to propose in section 2 an alternative method of counting. It takes advantage of the structure of the considered objects (a regular k-gon and the dihedral group of order k) and uses only the simplest concepts and techniques from Group Theory and Number Theory. It allows us to solve not only our problem (as it is done in section 3) but some other problems that can be reduced to the enumeration of all distinct labels on a regular polygon. We hope also that the proposed technique can be successfully applied to enumeration of other combinatorial objects.

2 Main result

Let $\mathcal{P} = P_1P_2 \ldots P_k$ be a fixed regular k-gon and C be a given finite subset of Z^k, where Z is the set of the integers. We shall consider an arbitrary element $x = (x_1, x_2, \ldots x_k)$ of C as a **label** on $\mathcal{P}$: for every $j = 1, 2, \ldots k$ the side P_jP_{j+1} is labelled by x_j (setting $P_{k+1} = P_1$). We shall assume that two labels $x' = \left(x'_1, x'_2, \ldots x'_k\right)$ and $x'' = \left(x''_1, x''_2, \ldots x''_k\right)$ are **distinct** if there exists no congruence Φ mapping $\mathcal{P}$ onto itself and such that $\Phi\left(x'\right) \stackrel{def}{=} \left(\Phi\left(x'_1\right), \Phi\left(x'_2\right), \ldots \Phi\left(x'_k\right)\right) = x''$.

Our main problem is to find $n(C)$ - the number of all distinct labels from C. For this purpose we shall present this number in an appropriate form. To do this, we shall assume that C is **symmetric** with respect to $\mathcal{P}$, i.e., for every congruence Φ mapping $\mathcal{P}$ onto itself from $x \in C$ it follows that $\Phi(x) \in C$, too.

Let us denote by $\mathcal{G}$ the group of all congruences of the regular k-gon and let $\mathcal{G}_1, \mathcal{G}_2, \ldots \mathcal{G}_m$ be all its subgroup. Let

$$C_i = \{x \mid x \in C, \ \forall \Phi \in \mathcal{G}_i : x = \Phi(x)\}$$

for every $i = 1, 2, \ldots m$ and

$$\mathcal{G}_x = \{\Phi \mid \Phi \in \mathcal{G}, \ x = \Phi(x)\}$$

for every $x \in C$, i.e., $\mathcal{G}_x$ is the maximal subgroup of $\mathcal{G}$ that maps x onto itself.

It is easily seen that every label x from C has $|\mathcal{G}| / |\mathcal{G}_x|$ **non-distinct** representatives in C - this is exactly the number of the coclasses in $\mathcal{G}$ generated by $\mathcal{G}_x$. Therefore we can rewrite $n(C)$ as follows

$$n(C) = \sum_{x \in C} \frac{|\mathcal{G}_x|}{|\mathcal{G}|}. \qquad (1)$$

On the other hand, let us suppose that $n(C)$ can have representation of the form

$$n(C) = \sum_{i=1}^{m} g_i |C_i|, \qquad (2)$$

where g_i, $i = 1, 2, \ldots m$ are some numbers (so far unknown).

Comparing the coefficients corresponding to an arbitrary $x \in C$ in the right sides of (1) and (2) we obtain

$$\frac{|\mathcal{G}_x|}{|\mathcal{G}|} = \sum_{\mathcal{G}_i \subseteq \mathcal{G}_x} g_i \qquad (3)$$

since all subgroups of $\mathcal{G}$ mapping x onto itself are just all subgroups of the maximal one which is $\mathcal{G}_x$. There are no more than m equations of the form (3) - the subgroup G_x in the left side is some of $\mathcal{G}_1, \mathcal{G}_2, \ldots \mathcal{G}_m$. Thus we obtain a linear system with unknown quantities $g_1, g_2, \ldots g_m$. If it has an unique solution (this would prove our assumption that $n(C)$ can be represented in the form (2)) that can be found explicitly and if this holds for all $|C_i|$, $i = 1, 2, \ldots m$ as well, then the value of $n(C)$ might be calculated from (2). In fact we can get these values by Möbius Inversion Technique (see e.g. [3], 12.7) but we prefer the simple way described further.

Let us note that the above considerations remain correct for an arbitrary object $\mathcal{P}$ with its group of symmetries $\mathcal{G}$. Now we shall concentrate on the case when $\mathcal{P}$ is a regular k-gon and $\mathcal{G}$ is the dihedral group of order k.

It is well known that $\mathcal{G}$ is generated by its elements ρ - a rotation at $2\pi/k$ and σ - symmetry with axis OA_1 , where O is the centre of $\mathcal{P}$- see Figure 2. So, we have

$$\mathcal{G} = \left\{e, \rho^1, \rho^2, \ldots \rho^{k-1}, \sigma, \rho\sigma, \rho^2\sigma, \ldots \rho^{k-1}\sigma\right\}$$

(see Figure 3).

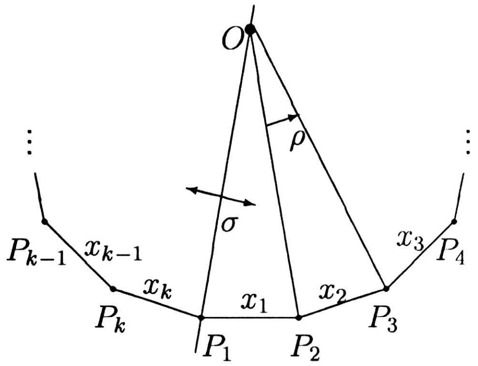

Figure 2

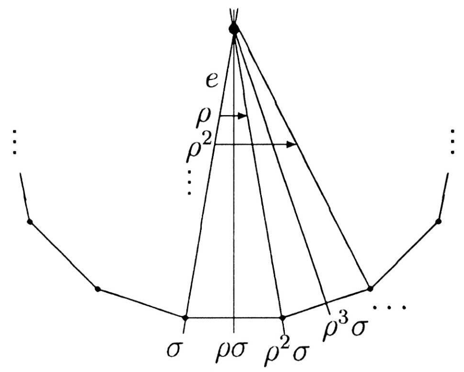

Figure 3

The subgroups of $\mathcal{G}$ are

$$\mathcal{R}_d = \left\{ e, \rho^d, \rho^{2d}, \dots \rho^{\left(\frac{k}{d}-1\right)d} \right\},$$

for all $d|k$ and

$$\mathcal{S}_{d,i} = \left\{ e, \rho^d, \rho^{2d}, \dots \rho^{\left(\frac{k}{d}-1\right)d}, \rho^i\sigma, \rho^{i+d}\sigma, \rho^{i+2d}\sigma, \dots \rho^{i+\left(\frac{k}{d}-1\right)d}\sigma \right\},$$

for all $d|k$, $0 \leq i \leq d-1$.

It is important to observe that $R_1 = S_{1,0} = G$, $R_2 = S_{2,1}$, $S_{2,0} = G$ (if $\mathcal{S}_{2,0}$ and $\mathcal{S}_{2,1}$ exist i.e. k is even) and all other R_d's and $S_{d,i}$'s are distinct. So, (2) becomes

$$n(C) = \sum_{d|k} r_d |R_d| + \sum_{d|k,\, d>2} \sum_{i=0}^{d-1} s_{d,i} |S_{d,i}|, \tag{4}$$

where

$$R_d = \{x \mid x \in C,\ \forall \Phi \in \mathcal{R}_d : x = \Phi(x)\}$$

for $d|k$ and

$$S_{d,i} = \{x \mid x \in C,\ \forall \Phi \in \mathcal{S}_{d,i} : x = \Phi(x)\}$$

for $d|k$, $0 \leq i \leq d-1$ and r_d's, $s_{d,i}$'s are unknown coefficients.

We shall describe for every subgroup of $\mathcal{G}$ its subgroups and its rank :

- for $\mathcal{R}_d$, $d|k$, they are $\mathcal{R}_j$, $d|j|k$. $|\mathcal{R}_d| = \frac{k}{d}$;
- for $\mathcal{S}_{d,i}$, $d|k$, $0 \leq i \leq d-1$, they are $\mathcal{R}_j$, $d|j|k$ and $\mathcal{S}_{j,i+dl}$, $d|j|k$, $0 \leq l \leq \frac{j}{d} - 1$. $|\mathcal{S}_{d,i}| = \frac{2k}{d}$

Thus the system of equations of the form (3) becomes

$$\sum_{d|j|k} r_j = \frac{1}{2d} \qquad d|k, d > 2 \tag{5}$$

$$\sum_{d|j|k} r_j + \sum_{d|j|k} \sum_{l=0}^{\frac{j}{d}-1} s_{j,i+dl} = \frac{1}{d} \qquad d|k,\ d>2,\ 0 \le i \le d-1 \tag{6}$$

$$\sum_{2|j|k} r_j + \sum_{2|j|k,\ j>2} \sum_{l=0}^{\frac{j}{2}-1} s_{j,1+2l} = \frac{1}{2} \qquad \text{iff } k \text{ is even} \tag{7}$$

$$\sum_{j|k} r_j + \sum_{j|k,\ j>2} \sum_{l=0}^{j-1} s_{j,l} = 1 \tag{8}$$

We shall prove the following lemma

Lemma 2.1 *The linear system (5), (6), (7), (8) has an unique solution* $r_d = \frac{1}{2k}\varphi\left(\frac{k}{d}\right)$ *for* $d|k$; $s_{k,i} = \frac{1}{2k}$ *for* $0 \le i \le k-1$ *and all other variables are zeros.*

Proof Let us remind the definition of the function φ : if $n = \prod_{i=1}^{m} p_i^{n_i}$ is the canonical representation of n then $\varphi(n) = \prod_{i=1}^{m}\left(p_i^{n_i} - p_i^{n_i-1}\right)$. We shall use also in the function $\tau(n) \stackrel{def}{=} \sum_{i=1}^{m} n_i$.

First we shall prove the uniqueness of the r_d's, $s_{d,i}$'s and that $s_{d,i} = 0$ for $2 < d < k$ using a limited induction on $\tau\left(\frac{k}{d}\right)$. If $\tau\left(\frac{k}{d}\right) = 0$ then $d = k$.From (5) we obtain $r_k = \frac{1}{2k}$ and from (6) follows that $s_{k,i} = \frac{1}{k} - r_k = \frac{1}{2k}$ for $0 \le i \le k-1$. Let us suppose that the assertion is true for all d such that $\tau\left(\frac{k}{d}\right) < i_0$. We shall prove it for every d_0 with $\tau\left(\frac{k}{d}\right) = i_0$. To do this we must consider the following three possible cases (having in mind that from $d_0|j,\ j > d_0$ it follows $\tau\left(\frac{k}{j}\right) < i_0$) :

1. (a) $d_0 > 2$. From (5), we have

$$r_{d_0} = \frac{1}{2d_0} - \sum_{d_0|j|k, j>d_0} r_j$$

and from (6) and (5)

$$s_{d_0,i} = \frac{1}{d_0} - \sum_{d_0|j|k} r_j - \sum_{d_0|j|k, d_0<j<k} \sum_{l=0}^{\frac{j}{d_0}-1} s_{j,i+d_0 l} - \sum_{l=0}^{\frac{k}{d_0}-1} s_{k,i+d_0 l} =$$

$$\frac{1}{d_0} - \frac{1}{2d_0} - \sum_{d_0|j|k,\ d_0<j<k} \sum_{l=0}^{\frac{j}{d_0}-1} s_{j,i+d_0 l} - \frac{k}{d_0} \cdot \frac{1}{2k} =$$

$$- \sum_{d_0|j|k, j>d_0>k} \sum_{l=0}^{\frac{j}{d_0}-1} s_{j,i+d_0 l} = 0.$$

2. (b) $d_0 = 2$ .This is possible iff k is even. From (7) we have

$$r_2 = \frac{1}{2} - \sum_{2|j|k,j>2} r_j - \sum_{2|j|k,2<j<k} \sum_{l=0}^{\frac{j}{2}-1} s_{j,1+2l} - \sum_{l=0}^{\frac{k}{2}-1} s_{k,1+2l} =$$

$$\frac{1}{2} - \sum_{2|j|k,j>2} r_j - 0 - \frac{k}{2} \cdot \frac{1}{2k} = \frac{1}{4} - \sum_{2|j|kj>2} r_j.$$

3. (c) $d_0 = 1$. From (8), we have

$$r_1 = 1 - \sum_{j|k,j>1} r_j - \sum_{j|k,2<j<k} \sum_{l=0}^{\frac{j}{2}-1} s_{j,1+2l} - \sum_{l=0}^{k-1} s_{k,l} =$$

$$1 - \sum_{j|k,j>1} r_j - 0 - k \cdot \frac{1}{2k} = \frac{1}{2} - \sum_{j|k,j>1} r_j.$$

Thus we have provednthat $s_{k,i} = \frac{1}{2k}$ for $0 \le i \le k-1$ and all other $s_{d,i}$'s are zeros. Adding the equations from the cases (b) and (c) to (5) we obtain also

$$\sum_{d|j|k} r_j = \frac{1}{2d} \tag{9}$$

for all $d|k$.

Now we shall show that $r_d = \frac{1}{2k}\varphi\left(\frac{k}{d}\right)$. Let us fix d and let $k = \prod_{i=1}^{m} p_i^{k_i}$ and $d = \prod_{i=1}^{m} p_i^{d_i}$ be the canonical representations of k and d. We define the function

$$f(z_1, z_2, \ldots z_m) \stackrel{def}{=} \prod_{i=1}^{m} \left(1 + \sum_{l=1}^{k_i - d_i} \left(p_i^l - p_i^{l-1}\right) z_i^l\right).$$

Rewriting $f(z_1, z_2, \ldots z_m)$ as a sum of monomes we see that the coefficient of the term $z_1^{l_1} z_2^{l_2} \ldots z_m^{l_m}$ is just $\varphi\left(p_1^{l_1} p_2^{l_2} \ldots p_m^{l_m}\right)$. We observe also that there exists bijection between integer numbers j such that $d|j|k$ (or equivalent $j = \prod_{i=1}^{m} p_i^{j_i}$, $d_i \le j_i \le k_i$, $1 \le i \le m$) and monomes of the form $z_1^{l_1} z_2^{l_2} \ldots z_m^{l_m}$, $0 \le l_i \le k_i - d_i$, $1 \le i \le m$ defined as

$$p_1^{j_1} p_2^{j_2} \ldots p_m^{j_m} \longleftrightarrow z_1^{k_1 - j_1} z_2^{k_2 - j_2} \ldots z_m^{k_m - j_m}.$$

So, we have

$$f(1,1,\dots 1) = \sum_{d_i \le j_i \le k_i,\, 1 \le i \le m} \varphi\left(p_1^{k_1-j_1} p_2^{k_2-j_2} \dots p_m^{k_m-j_m}\right) = \sum_{d|j|k} \varphi\left(\frac{k}{j}\right).$$

On the other hand

$$f(1,1,\dots 1) = \prod_{i=1}^{m} \left(1 + \sum_{l=1}^{k_i-d_i} \left(p_i^l - p_i^{l-1}\right)\right) = \prod_{i=1}^{m} p_i^{k_i-d_i} = \frac{k}{d}$$

and consequently we obtain that

$$\sum_{d|j|k} \varphi\left(\frac{k}{j}\right) = \frac{k}{d} \quad \text{i.e.} \quad \sum_{d|j|k} \frac{1}{2k}\varphi\left(\frac{k}{j}\right) = \frac{1}{2d}$$

for every d, $d|k$. But the r_j's satisfying the equations (9) are unique, so it follows that $r_d = \frac{1}{2k}\varphi\left(\frac{k}{d}\right)$.□

Now we are able to prove our main result.

Theorem 2.1 *Let the set $C \subseteq Z_+^k$ define a label over the sides of the regular k-gon $\mathcal{P}$ and C be symmetric with respect to $\mathcal{P}$. The number of all distinct elements from C is*

$$n(C) = \frac{1}{2k}\sum_{d|k} \varphi\left(\frac{d}{k}\right)|R_d| + \begin{cases} \frac{1}{2}|S_1| & k \text{ is odd} \\ \frac{1}{4}(|S_0| + |S_1|) & k \text{ is even} \end{cases},$$

where

$$R_d = \{x \in C \mid x = (x_1, x_2, \dots x_d, x_1, x_2, \dots x_d, \dots\dots\dots x_1, x_2, \dots x_d)\}$$

for all $d|k$ and

$$S_0 = \{x \in C \mid x = (x_1, x_2, x_3, \dots x_3, x_2, x_1)\}$$

$$S_1 = \{x \in C \mid x = (x_1, x_2, x_3, x_4 \dots x_4, x_3, x_2)\}.$$

Proof From lemma 2.1 and (4) we obtain

$$n(C) = \frac{1}{2k}\sum_{d|k} \varphi\left(\frac{k}{d}\right)|R_d| + \frac{1}{2k}\sum_{i=0}^{k-1} |S_{k,i}|.$$

So let us consider the $S_{k,i}$'s

$$S_{k,i} = \left\{x \mid x \in C,\ x = \rho^i \sigma(x)\right\}$$

i.e.

$$S_{k,i} = \{x \mid x \in C,\ (x_1, \dots x_i, x_{i+1}, \dots x_k) = (x_i, \dots x_1, x_k, \dots x_{i+1})\}.$$

We have one of the following two cases :

1. (a) k is odd. Let us suppose that i is odd, too, and observe that $x \in S_{k,i}$ iff $\rho^{-\frac{i+1}{2}}(x) \in S_{k,1}$ i.e. $\rho^{-\frac{i+1}{2}}$ defines a bijection between $S_1 = S_{k,1}$ and $S_{k,i}$.If i is even this bijection is defined by $\rho^{\frac{k+i+1}{2}}$. But then we have

$$\frac{1}{2k}\sum_{i=0}^{k-1}|S_{k,i}| = \frac{1}{2k}\cdot k\cdot|S_{k,1}| = \frac{1}{2}|S_1|.$$

2. (b) k is even. In this case if i is even then $\rho^{-\frac{i}{2}}$ defines a bijection between $S_0 = S_{k,0}$ and $S_{k,i}$; if i is odd then $\rho^{-\frac{i+1}{2}}$ defines a bijection between $S_1 = S_{k,1}$ and $S_{k,i}$. Thus we obtain

$$\frac{1}{2k}\sum_{i=0}^{k-1}|S_{k,i}| = \frac{1}{2k}\sum_{0\le i\le k-1,\ i\text{ is even}}|S_{k,i}| + \frac{1}{2k}\sum_{0\le i\le k-1,\ i\text{ is odd}}|S_{k,i}| =$$

$$\frac{1}{2k}\cdot\frac{k}{2}\cdot|S_{k,0}| + \frac{1}{2k}\cdot\frac{k}{2}\cdot|S_{k,1}| = \frac{1}{4}\left(|S_0| + |S_1|\right).$$

□

3 The number of integer k-gons, having perimeter n

According to theorem 2.1, we need the values of $|S_0|$, $|S_1|$ and $|R_d|$ for all d, $d|k$, where

$$C = \left\{x \mid x \in Z_+^k,\ \sum_{i=1}^{k} x_i = n,\ x_i < n - x_i \text{ for all } i = 1,2,\ldots n\right\}$$

(here Z_+ is the set of the positive integers).

We shall use the following two elementary assertions :

- (a) The number of all vectors $x \in Z_+^k$ such that $\sum_{i=1}^{k} x_i = n$ is $\binom{n-1}{k-1}$.
- (b) $\sum_{j=a}^{b}\binom{j}{k} = \binom{b+1}{k+1} - \binom{a}{k+1}$ holds for every $k \ge 0$ and $b \ge a-1$.

Let d is such that $d|k$, $d < k$. We have

$$R_d = \{x \mid x \in Z_+^k,\ x = (x_1,\ldots x_d, x_1,\ldots x_d,\ldots\ldots\ldots x_1,\ldots x_d),\ \frac{k}{d}\sum_{i=1}^{d} x_i = n\}$$

(note that the condition $x_i < n - x_i$ for all i follows from the others). Using (a), we obtain

$$|R_d| = \begin{cases} \binom{\frac{nd}{k}-1}{d-1}, & \frac{k}{d}\mid n \\ 0, & \text{otherwise} \end{cases} .$$

Let $d = k$. In this case $R_k = C$ and

$$C = \left\{x \mid x \in Z_+^k,\ \sum_{i=1}^{k} x_i = n\right\} \setminus \bigcup_{p=\lceil \frac{n}{2}\rceil}^{n-k+1} \left\{x \mid x \in Z_+^k,\ \sum_{i=1}^{k} x_i = n,\ \exists j : x_j = p\right\}.$$

We observe that all sets in the union are disjoint and that the index j can take just k values - from 1 to k. So, we have (using (a) and (b))

$$|R_k| = \binom{n-1}{k-1} - \sum_{p=\lceil \frac{n}{2}\rceil}^{n-k+1} k\binom{n-p-1}{k-2} = \binom{n-1}{k-1} - k\binom{\lfloor \frac{n}{2}\rfloor}{k-1}.$$

Let us compute $|S_0|$. We shall consider two cases :

- k is even. Then

$$S_0 = \left\{x \mid x \in Z_+^k,\ x = \left(x_1, x_2, \ldots, x_{\frac{k}{2}}, x_{\frac{k}{2}}, \ldots x_2, x_1\right),\ 2\sum_{i=1}^{\frac{k}{2}} x_i = n\right\}$$

and it is easy to see the bijection between $|S_0|$ and $\left|R_{\frac{k}{2}}\right|$, so

$$|S_0| = \begin{cases} \binom{\frac{n}{2}-1}{\frac{k}{2}-1}, & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases} .$$

- k is odd. In this case

$$S_0 = \left\{x \in Z_+^k \mid x = \left(x_1, \ldots x_{\frac{k+1}{2}}, \ldots x_1\right),\ x_{\frac{k+1}{2}} + 2\sum_{i=1}^{\frac{k-1}{2}} x_i = n,\ x_{\frac{k+1}{2}} < n - x_{\frac{k+1}{2}}\right\}$$

If we denote $p = \sum_{i=1}^{\frac{k-1}{2}} x_i$ then from $x \in Z_+^k$ it follows that $p \geq \frac{k-1}{2}$ and $n - 2p \geq 1$. From $x_{\frac{k+1}{2}} < n - x_{\frac{k+1}{2}}$ follows $4p > n$. So, we obtain

$$|S_0| = \sum_{p=\max\left(\lceil \frac{n+1}{4}\rceil, \frac{k-1}{2}\right)}^{\lfloor \frac{n-1}{2}\rfloor} \binom{p-1}{\frac{k-3}{2}} = \binom{\lfloor \frac{n-1}{2}\rfloor}{\frac{k-1}{2}} - \binom{\lceil \frac{n-3}{4}\rceil}{\frac{k-1}{2}}$$

(we have used (a) and (b)).

Finally we shall compute $|S_1|$. Again we shall do this in two cases :

- k is odd. We have

$$S_1 = \left\{ x \in Z_+^k \mid x = \left(x_1, x_2 \ldots x_{\frac{k+1}{2}}, x_{\frac{k+1}{2}} \ldots x_2\right),\ x_1 + 2\sum_{i=2}^{\frac{k+1}{2}} x_i = n,\ x_1 < n - x_1 \right\}$$

and the bijection between S_1 in this case and S_0 when k is odd is obvious, so

$$|S_1| = \binom{\left\lfloor \frac{n-1}{2} \right\rfloor}{\frac{k-1}{2}} - \binom{\left\lceil \frac{n-3}{4} \right\rceil}{\frac{k-1}{2}}.$$

- k is even. In this case

$$S_1 = \left\{ x \mid x \in Z_+^k,\ x = \left(x_1, x_2 \ldots x_{\frac{k}{2}-1} x_{\frac{k}{2}}, x_{\frac{k}{2}-1} \ldots x_2\right), \right.$$
$$\left. x_1 + x_{\frac{k}{2}} + 2\textstyle\sum_{i=2}^{\frac{k}{2}-1} x_i = n,\ x_1 < n - x_1,\ x_{\frac{k}{2}} < n - x_{\frac{k}{2}} \right\}.$$

Let us denote $p = \sum_{i=2}^{\frac{k}{2}-1} x_i$, $q = x_{\frac{k}{2}}$. From $x \in Z_+^k$ it follows that $p \geq \frac{k}{2} - 1$, $q \geq 1$ and $n - 2p - q \geq 1$. From $x_1 < n - x_1$ and $x_{\frac{k}{2}} < n - x_{\frac{k}{2}}$ it follows that $2q > n - 4p$ and $2q < n$. Thus we have

$$|S_1| = \sum_{p=\frac{k}{2}-1}^{\lfloor \frac{n}{2} \rfloor - 1} \binom{p-1}{\frac{k}{2}-2} q =$$

$$\sum_{p=\frac{k}{2}-1}^{\lfloor \frac{n}{2} \rfloor - 1} \binom{p-1}{\frac{k}{2}-2} \left(\min\left(n - 2p - 1, \left\lfloor \frac{n-1}{2} \right\rfloor\right) - \max\left(1, \left\lceil \frac{n+1}{2} \right\rceil - 2p\right) + 1 \right)$$

and we obtain (after simplification, using (b))

$$|S_1| = \left(\frac{k(n-1)}{2\left\lfloor \frac{n}{2} \right\rfloor} - (k-2) \right) \binom{\left\lfloor \frac{n}{2} \right\rfloor}{\frac{k}{2}} + 2(k-2) \binom{\left\lceil \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor \right\rceil}{\frac{k}{2}} - 2\left\lfloor \frac{n}{2} \right\rfloor \binom{\left\lceil \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor \right\rceil - 1}{\frac{k}{2} - 1}$$

This completes our solution. Finally, we shall mention only that in the case when k is and prime number, $T(k,n)$ accepts a very simple form :

$$T(k,n) = \frac{1}{2k} \left(\binom{n-1}{k-1} - k \binom{\left\lfloor \frac{n}{2} \right\rfloor}{k-1} \right) + \frac{1}{2} \left(\binom{\left\lfloor \frac{n-1}{2} \right\rfloor}{\frac{k-1}{2}} - \binom{\left\lceil \frac{n-3}{4} \right\rceil}{\frac{k-1}{2}} \right) + \begin{cases} \frac{k-1}{2k}, & k \mid n \\ 0, & \text{otherwise} \end{cases}.$$

4 Conclusions and open problems

In our paper we have obtained a closed form of the number of all distinct integer labels on the sides of the regular k-gon satisfying a given condition. We have proved our result independently from the Pólya's Theory of Counting and we have used only elementary concepts and techniques from Group Theory and Number Theory.

Finally we shall set two open problems having an analogical formulation. We hope that they can be solved by use of similar methods as in the present paper.

1. How many are there "distinct" polyhedrals, having k vertices and integer edges with sum n?

and

2. How many are there "distinct" integer triangulations of a k − gon with sum n?

References

[1] N. G. DE BRUJN, *A survey of generaliztions of Pólya's enumeration theorem,* Nieuv Archief voor Wiskunde, (2), 19 (1971), pp.89-112.

[2] N. G. DE BRUJN, *Pólya's Theory of Counting,* ch. 5 in Applied Combinatorial Mathematics, E. F. Beckenbach, ed. (1964), pp.144-184.

[3] P. J. CAMERON, *Combinatorics: Topics, Techniques, Algorithms,* Cambridge University Press, 1994.

[4] C. W. CURTIS and I. REINER, *Representation theory of finite groups and associative algebras,* Interscience Publishers, a division of John Wiley & Sons, New York, London, 1962.

[5] J. JORDAN, R. WALCH and W. WISNER, *Triangles with integer sides,* Amer. Math. Monthly, (8), 86 (1979), pp. 686-689.

[6] G. PÓLYA, *Kombinatorishe Anzahlbestimmugen für Gruppen, Graphen und chemische Verbindungen,* Acta Mathematica, 68 (1937), pp. 145-254.