NNTDM 6 (2000) 2, 64-68

ON A FORMULA RELATED TO THE $n-$TH PARTIAL SUM OF THE HARMONIC SERIES

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The harmonic series

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+...$$

has a remarkable property: for $n>1$ its $n-$th partial sum

$$\frac{1}{1}+\frac{1}{2}+...+\frac{1}{n}$$

is never integer. We may consider the more general series

$$\frac{1}{1.p+1}+\frac{1}{2.p+1}+\frac{1}{3.p+1}+...$$

(p is a positive real number) for which the above property is still valid, at least for the case $p=1,2,3,...$, with respect to its partial sum

$$1+\frac{1}{1.p+1}+\frac{1}{2.p+1}+...+\frac{1}{(n-1).p+1}.$$

Therefore, the question: *"what does the integer part of the last partial sum equal to" is reasonable (cf. [1,2]).*

The present paper gives the answer of this question (see [3]), when n denotes the integer part of the numbers

$$\frac{e^{kp}-1}{p}+1 \quad (k=1,2,3,...).$$

A very interesting fact is that the integer part of the mentioned partial sum, for such values of n, does not depend on p, but only on k.

below one may see the complete investigation.

Firstly, we shall introduce the following denotations: $\mathcal{N}$ *- the set of all positive integers;* $\mathcal{R}^+$ *- the set of all positive real numbers;* $e=\lim\limits_{m\to\infty}(1+\frac{1}{m})^m=2.718...$; H_n *- the n-th partial sum of the harmonic resies;* $[x]$ *is the integer part of* x*, i.e., the greatest integer* y *for which* $y\leq x$*, when* $x\in\mathcal{R}^+$ *or* $x=0$*;* $\{x\}=x-[x]$.

Definition 1. *For $p \in \mathcal{R}^+$ and $n \in \mathcal{N}$ we introduce $H_n(p)$ by*

$$H_n(p) = \sum_{i=0}^{n-1} \frac{1}{i.p+1}. \qquad (1)$$

Remark 1. *Obviously, $H_n(1)$ coincides with H_n and for $n \geq 3$ we have:*

$$H_n(2) = 1 + \frac{1}{3} + \frac{1}{5} + ... + \frac{1}{2n-1}; \qquad (2)$$

$$H_n(3) = 1 + \frac{1}{4} + \frac{1}{7} + ... + \frac{1}{3n-2}; \qquad (3)$$

$$H_n(4) = 1 + \frac{1}{5} + \frac{1}{9} + ... + \frac{1}{4n-3}; \qquad (4)$$

and etc.

Definition 2. *For $k \in \mathcal{N}$ and $p \in \mathcal{R}^+$ we introduce $n_k(p)$ by*

$$n_k(p) = [\frac{e^{kp}-1}{p}] + 1. \qquad (5)$$

It is not difficult to prove the following

LEMMA *The number $n_k(p)$ admits the representation*

$$n_k(p) = \begin{cases} [\frac{e^{kp}}{p}] - [\frac{1}{p}], & for\ \{\frac{e^{kp}}{p}\} < \{\frac{1}{p}\} \\ \\ [\frac{e^{kp}}{p}] - [\frac{1}{p}] + 1, & for\ \{\frac{e^{kp}}{p}\} \geq \{\frac{1}{p}\} \end{cases} \qquad (6)$$

Especially, for $p \geq 1$ (6) takes the form

$$n_k(p) = \begin{cases} [\frac{e^{kp}}{p}], & for\ \{\frac{e^{kp}}{p}\} < \{\frac{1}{p}\} \\ \\ [\frac{e^{kp}}{p}] + 1, & for\ \{\frac{e^{kp}}{p}\} \geq \{\frac{1}{p}\} \end{cases} \qquad (7)$$

Hence

$$n_k(1) = [e^k]. \qquad (8)$$

The main result of the paper is

THEOREM *For every $k \in \mathcal{N}$ and $p \in \mathcal{R}^+$ the identity*

$$[1 + \frac{1}{1.p+1} + ... + \frac{1}{[\frac{e^{kp}-1}{p}].p+1}] = k, \qquad (9)$$

i.e.,

$$[H^{(p)}_{n_k(p)}] = k,$$

holds.

Remark 2. *The theorem shows that the left hand side of (9) does not depend on p, but only on k, which is very unexpected.*

Proof of the Theorem *Let $p \in \mathcal{R}^+$ be arbitrary chosen.*

1. Using that

$$f(x) = (1 + \frac{1}{x})^x$$

is an increasing function on $(0, +\infty)$ and the fact that

$$\lim_{x \to \infty} f(x) = e$$

we obtain

$$(1 + \frac{1}{m + \frac{1}{p}})^{m + \frac{1}{p}} < e$$

for $m = 0, 1, 2, \ldots$. Hence

$$\ln(1 + \frac{1}{m + \frac{1}{p}}) < \frac{1}{m + \frac{1}{p}}$$

The last inequality yields

$$\frac{1}{p}.(\ln(m + 1 + \frac{1}{p}).\ln(m + \frac{1}{p})) < \frac{1}{m.p + 1}. \quad (10)$$

Let $n \in \mathcal{N}$. We put in (10) $m = 0, 1, 2, \ldots, n - 1$ and add the corresponding inequalities to obtain

$$\frac{1}{p}.\ln(n.p + 1) < H_n(p). \quad (11)$$

2. Using that

$$g(x) = (1 + \frac{1}{x - 1})^x$$

is a decreasing function on $(1, +\infty)$ and the fact that

$$\lim_{x \to \infty} g(x) = e$$

we obtain

$$e < (1 + \frac{1}{m - 1 + \frac{1}{p}})^{m + \frac{1}{p}}.$$

for $m = 1, 2, \ldots$. Hence

$$\frac{1}{m + \frac{1}{p}} < \ln(1 + \frac{1}{m - 1 + \frac{1}{p}}).$$

The last inequality yields

$$\frac{1}{m.p+1} < \frac{1}{p}.(\ln(m+\frac{1}{p}) - \ln(m-1+\frac{1}{p})). \tag{12}$$

Let $n \in \mathcal{N}$, $n > 1$. *We put in (12)* $m = 1, 2, ..., n-1$ *and add the corresponding inequalities to obtain*

$$\sum_{m=1}^{n-1} \frac{1}{m.p+1} < \frac{1}{p}.\ln(n.p-p+1)$$

Hence

$$H_n(p) < 1 + \frac{1}{p}.\ln(n.p-p+1). \tag{13}$$

3. For $n > 1$ *(11) and (13) imply*

$$\frac{1}{p}.\ln(n.p+1) < H_n(p) < 1 + \frac{1}{p}.\ln(n.p-p+1). \tag{14}$$

Let $k \in \mathcal{N}$ *be fixed and* $n_k(p)$ *be given by (5) (instead of (5) one may prefer (6), or (7) when* $p \geq 1$*). Then it is easy to check the inequalities:*

$$k < \frac{1}{p}.\ln(p.n_k(p)+1); \tag{15}$$

$$1 + \frac{1}{p}.\ln(p.n_k(p)-p+1) \leq k+1. \tag{16}$$

But obviously we have

$$n_k(p) > 1, \tag{17}$$

because of the inequality

$$e^{k.p} > p+1$$

and (5). Then putting into (14) $n_k(p)$ *instead of* n *and using (15) and (16) we finally obtain*

$$k < H_{n_k(p)}(p) < k+1. \tag{18}$$

But (18) means that (9) is true and the theorem is proved.

Corollary. *For every* $k \in \mathcal{N}$ *the identity*

$$[H_{[e^k]}] = k \tag{19}$$

holds.

Indeed

$$H_n = H_n(1)$$

(see Remark 1). Therefore (19) follows from (9) putting there $p = 1$, because of (8).

In [1] another result related to the $n-th$ partial sum of the harmonic series is proposed:

$$[H_{V(k)]}] = k,$$

where

$$V(k) = \sum_{i=0}^{k-1} e^i.$$

Finally we need two observations:

Observation 1. *If $p = 0$ we define the left hand side of (9) as*

$$[\lim_{p \to +0} (1 + \frac{1}{1.p+1} + \ldots + \frac{1}{[\frac{e^{kp}-1}{p})].p+1}]$$

i.e., as

$$[\lim_{p \to +0} (1 + \frac{1}{1.p+1} + \ldots + \frac{1}{kp+1})]$$

since

$$\lim_{p \to +0} \frac{e^{kp}-1}{p} = k.$$

Hence the left hand side of (9) equals to $k+1$ when $p = 0$.

Observation 2. *Here we put the question:* when (16) is a pure inequality?

It is easy to see that (16) is an equality if and only if the condition

$$\frac{e^{kp}-1}{p} \in \mathcal{N} \qquad (20)$$

is satisfied. But if p is an algebraic number, then the well known Lindemann's theorem ([2], Theorem 10.1) shows that (20) is not possible. Hence (16) is certainly a pure inequality if p is an algebraic number.

References:

[1] Atanassov K., L. Asenova, Problem 142, Physico-math. J. Bulg. Acad. of Sci., 21 (54), 1978, No. 4, 308.

[2] Atanassov K., Remark on the harmonic series. Comptes Rendus de l'Academie bulgare des Sciences, Tome 40, 1987, No. 5, 25-28.

[3] Feldman N., Hilbert's seventh problem. Moskow, Nauka, 1982 (in Russian).