NNTDM 6 (2000) 2, 45-55

ON THE ONE-SIDED ORDERABILITY FOR SEMIGROUPS

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Abstract In the papers of Merlier [5] and Saito [7] there are given some necessary and sufficient conditions on the linear orderability of the bands. Similar questions on semigroups are treated by Jordjev, Todorov [4] and Todorov [9]. In the last paper there are studied for the first time the one-sided orderable semigroups. We consideraby enlarge the last studies, giving conditions under which a given semigroup should not be linearly orderable, (being perhaps left or right stable orderable) and conditions when a semigroup is not one sided orderable, or when it is no-one-sided orderable.

Introduction

As it is known, any finite non-trivial group is not linearly (totally) orderable. Meanwhile, as it is shown by Gabovich [3] and by Zibina [10], there are (also finite) semigroups which are orderable relatively to every order in their sets. The question naturally arised in this situation "when a (finite) semigroup is totally orderable" was formulated for the first time time by Schein in [8]. Solutions of this problem in bands, formulated in terms of subbands, are given in the papers of Merlier [5] and Saito [7]. Gabovich [3], Jordjev and Todorov [4] formulate criterias on the linear ordering in groupoids and semigroups, respectively, formulated in set-theoritical terms. Todorov in [9] presents definite classes of semigroups by the generating elements and defining relations and prove that they are linearly orderable. In that paper there are studied for the first time the one sided orderable semigroups.
There exist semigroups which are not linear orderable but admit any left or right stable order. One of them is the semigroup S_3[7]:

$$\begin{matrix} 1 & 3 & 3 & 3 \\ 3 & 2 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4. \end{matrix}$$

This semigroup admits orders which are left stable (for example the orders $2 < 3 < 4 < 1$ and $2 < 4 < 3 < 1$), orders which are right stable (like $1 < 3 < 2 < 4$ and $4 < 1 < 3 < 2$) as well as orders which are neither left nor right stable. One of them is $2 < 1 < 3 < 4$.

The above studies and the last example arise the questions on finding conditions that a semigroup shuld be:

- no (two-sided) stable orderable.
- no left [right] stable orderable.
- no-one-sided (that is, neither left nor right) stable orderable.

Our approach to the problem consists just in finding solutions of these questions. There are found conditions that a semigroup should not be (two-sided) orderable, (being perhaps left or right stable orderable) (Theorem 1, Corollary 2, Theorem 2 a), Theorem 3 a)) and conditions when a semigroup is not one sided orderable (Theorem 1, Corollary 1, Theorem 2 b), Theorem 3 b)), or when it is no-one-sided orderable (Proposition 3).
The examples used to illustrate the obtained conditions and the created situations are mainly chosen among the semigroups $S_1 - S_{11}$[7] and $S_1 - S_4$[5] that play a decisive role in the bands' non-orderability.
All the non-explained terms and notations are taken from Gabovich [3] and Clifford and Preston [2].

The author thanks K. J. Todorov for his guidance and his very helpful suggestions throughout the preparation of this paper.

Preliminaries

In a semigroup S we shall consider the left [right] inner translations λ_a [ρ_a] defined by $x\lambda_a = ax$ [$x\rho_a = xa$]. Each of them determines an equivalence relation in S: two elements x and y of S are considerd left [right] equivalent if the equality

$$x\lambda_a = y\lambda_a \ [x\rho_a = y\rho_a]$$

holds. The equivalence class of x determined by the left inner translation λ_a is denoted by $[x\lambda_a]$, that is

$$[x\lambda_a] = \{y \in S \ / \ ax = ay\}.$$

Analogously, by $[x\rho_a]$ are denoted the equivalence classes determined by the right inner translations.
We shall use throughout the paper the following familiar notation.
The semigroup S is called *left orderable* if there exists any linear order $<$ in the set S which is left stable, i.e. such that $x < y$ implies $ax \leq ay$ for every $x, y, a \in S$.
Analogously is defined the *right orderability.*
The semigroup S is called *orderable* if it exist any linear order $<$ in the set S which is left stable and right stable.
Throughtout the paper the orders of the semigroups will be studied up to the dual o-isomorphic ones.
Let $(S, <)$ be an ordered set.The subset U is called a *convex* subset of S if

$$x \leq z \leq y \text{ for every } x, y \in U \text{ and } z \in S \text{ imply } z \in U.$$

As it is known (Gabovich [3], Zibina [10]), there exist classes of (finite) semigroups where every order is stable. Moreover, there exist classes of (finite) semigroups where the stability of the orders depends on the semigroups' cardinal.

Proposition 1 *Let B_n be the band generated by the elements $a_1, a_2, \ldots, a_n$, where*

$$a_i a_j = \begin{cases} a_i & \text{for } i = j \\ a_1 & \text{for } i \neq j. \end{cases}$$

Then

1. *Every order in the semigroup B_2 is stable;*

2. *In B_3 there exist orders which are two-sided stable, and orders which are neither left nor right stable;*

3. *Every order in B_n is no-one-sided stable.*

Proof.

1. Immediately (Gabovich [3], Zibina [10]);

2. The band B_3 is commutative, the order $a_3 < a_1 < a_2$ is stable, and the order $a_3 < a_2 < a_1$ is not stable.

3. Every one of the semigroups B_n contains the semigroup B_4 ($= S_7$[7]) which is a no-one-sided stable one

■

We note that the bands B_n are left regular ones ($efe = ef$ for every $e, f \in B_n$)
Also, there are classes of semigroups which are only one-sided orderable:

Proposition 2 *A rectagular band is one sided orderable, but not orderable.*

Proof. Let the rectangular band

$$I \times \Lambda$$

be ordered by the cartesian product of the orders $\leq_I$ and $\leq_\Lambda$:

$$\leq \;=\; \leq_I \times \leq_\Lambda$$

and let

$$(i_1, \lambda_1) \leq (i_2, \lambda_2) \qquad (*)$$

This means that

1. $i_1 <_I i_2$, or

2. $i_1 = i_2$ but $\lambda_1 \leq_\Lambda \lambda_2$.

Multiplying both sides of the inequality $(*)$ on the right by the element $(j, \mu) \in I \times \Lambda$ we get the obvious inequality

$$(i_1, \mu) \leq (i_2, \mu)$$

which means that the order $\leq$ is a right stable one.
Getting the components of the inequality $(*)$ such that

$$i_1 <_I i_2, \quad \lambda_1 >_\lambda \lambda_2$$

and multiplying its both sides on the left by $(j, \mu) \in I \times \Lambda$, we get

$$(j, \mu)(i_1, \lambda_1) = (j, \lambda_1) > (j, \lambda_2) = (j, \mu)(i_2, \lambda_2)$$

which means that the order $\leq$ in $I \times \Lambda$ can not be a two-sided stable one. ■

Main Results

We formulate the obtained result in the following

Lemma 1 *If the semigroup S is left [right] ordered, then the equivalence classes determined by the left [right] inner translations for all a in S are convex classes.* ■

Proof. Let x, y be two left equivalent elements of a left ordered semigroup S, that is

$$ax = ay \tag{1}$$

for any a in S, and z be an element of S such that

$$x \leq z \leq y.$$

Then, by the left stability of the order of S:

$$ax \leq az \leq ay$$

and by (??) :

$$ax = az = ay.$$

That means that the element z is in the same equivalence class with x and y. ■

We note that, in connection with the following study, the above mentioned semigroup B_4 admits a system of inner left translation classes:

$$C_1 = \{a_1, a_3, a_4\}, C_2 = \{a_1, a_2, a_4\}, C_3 = \{a_1, a_2, a_3\}$$

such that every one of them contains two of the the elements a_2, a_3, a_4 and not the third one.

Theorem 1 *If the semigroup S has n ($n \geq 3$) equivalence classes determined by its inner translations $C_1, C_2, \ldots, C_n$ and n elements $x_1, x_2, \ldots, x_n$ such that*

$$\{x_i, x_{i+1(mod\ n)}\} \subseteq C_i,\ x_{i-1(mod\ n)} \notin C_i,\ x_{i+2(mod\ n)} \notin C_i,\quad \forall i = 1, 2, \cdots, n \qquad (2)$$

then S is not orderable.
Moreover, if the classes C_i are all determined by left [right] inner translations of S, then S is not left [right] orderable.

Proof. Let us suppose that the semigroup S has any linear order $\leq$, stable in respect of its operation. Then each one of the equivalence classes C_i is convex. Let us also suppose, without any loss of generality, that

$$x_1 < x_2$$

We shall show that

$$x_2 < x_3.$$

Really, if $x_1 < x_3 < x_2$, then the convexity of C_1 yields $x_3 \in C_1$, which contradicts the last of the conditions (??) for $i = 1$. Also, if $x_3 < x_1 < x_2$, then the convexity if C_2 yields $x_1 \in C_2$, which contradicts the second of the conditions (??). So, $x_2 < x_3$.
Let us now suppose the sequence of inequalities:

$$x_1 < x_2 < \cdots < x_i \qquad (2 < i < n)$$

The element x_{i+1} can not be in the position

$$x_{i-1} < x_{i+1} < x_i$$

because of the convexity of C_{i-1} and the third if the conditions (??); also it can not be in the position

$$x_{i+1} < x_{i-1} < x_i$$

because of the convexity of C_i and the second of the conditions (??).
So, for any $i < n$, we have $x_i < x_{i+1}$, and in the same way one can verify that this result holds even when the arithmetic of the indexes is made in the ring of remainders modulo n $\mathbf{Z}_n = \mathbf{Z}/(n)$, that is, for $i = n$. That means that

$$x_n < x_{n+1(mod\ n)} = x_1.$$

The obtained sequence of inequalities

$$x_1 < x_2 < \cdots < x_n < x_1$$

is clearly a contradiction, so the order $\leq$ in the semigroup S can not be stable.
If all the classes C_i are determined by left [right] inner tranlations of S, then in the same way one can easily prove that the semigroup S can not be left [right] orderable. ■

For $n = 3$ and $n = 4$, this theorem gives the following more easily appliable conditions.

Corollary 1 *If the semigroup S has one of the following properties:*

$$L1)\quad \exists (x,y,z,t,u,v)\in S^6: \quad \begin{array}{ccccc} xt & = & xu \neq & xv \\ yt & \neq & yu = & yv \\ zt & = & zv \neq & zu \end{array}$$

$$L2)\quad \exists (x,y,z,t,u,v,w,s)\in S^8: \quad \begin{array}{ccccc} ut & \neq & ux=uy & \neq & uz \\ vx & \neq & vy=vz & \neq & vt \\ wy & \neq & wz=wt & \neq & wx \\ sz & \neq & st=sx & \neq & sy \end{array}$$

$$L3)\quad \exists (x,y,z,t,u,v,w,s)\in S^8: \quad \begin{array}{ccccc} ux & =uy & \neq & uz & =ut \\ vy & =vz & \neq & vt & =vx \end{array}$$

then it is not left orderable.

Proof. The conditions of Theorem **??** are fulfilled for:
L1) $n=3$, the elements $x_1=t, x_2=u, x_3=v$ and the classes $C_1\in S/\lambda_x$, $C_2\in S/\lambda_y$, $C_3\in S/\lambda_z$;
L2) $n=4, x_1=x,\ x_2=y,\ x_3=z,\ x_4=t,\quad C_1\in S/\lambda_u,\ C_2\in S/\lambda_v,\ C_3\in S/\lambda_w, C_4\in S/\lambda_s$.
The condition L3) is condition L2) for $w=4$ and $s=v$. ■

The following semigroups (S_7[7] and S_8[7]):

1	4	4	4
4	2	4	4
4	4	3	4
4	4	4	4

1	4	4	4	4	4
5	2	5	5	5	5
6	6	3	6	6	6
4	4	4	4	4	4
5	5	5	5	5	5
6	6	6	6	6	6

are typical examples where the condition L1) is satisfied. It holds (in both of them) for $x=v=1,\ y=t=2,\ z=u=3$

By the condition L3) the semigroup S_3[7]:

1	2	3	4
2	2	4	4
3	4	3	4
4	4	4	4

is not left orderable, taking $y=1,\ x=u=2,\ z=v=3,\ t=4$.

Corollary 2 *If the semigroup S has one of the following properties:*

$$T1)\quad \exists (x,y,z,t,u,v)\in S^6: \quad \begin{array}{ccccc} xt & = & xu & \neq & xv \\ yt & \neq & yu & = & yv \\ tz & = & vz & \neq & uz \end{array}$$

$$T2)\quad \exists (x,y,z,t,u,v,w,s)\in S^8: \quad \begin{array}{ccccccc} ux & = & uy & \neq & uz & = & ut \\ yv & = & zv & \neq & tv & = & xv \end{array}$$

$$T3)\quad \exists(x,y,z,t,u,v,w,s)\in S^8: \quad \begin{array}{ccccccc} ut & \neq & ux & = & uy & \neq & uz \\ vx & \neq & vy & = & vz & \neq & vt \\ yw & \neq & zw & = & tw & \neq & xw \\ zs & \neq & ts & = & xs & \neq & ys \end{array}$$

$$T4)\quad \exists(x,y,z,t,u,v,w,s)\in S^8: \quad \begin{array}{ccccccc} ut & \neq & ux & = & uy & \neq & uz \\ vx & \neq & vy & = & vz & \neq & vt \\ wy & \neq & wz & = & wt & \neq & wx \\ zs & \neq & ts & = & xs & \neq & ys \end{array}$$

then it is not orderable.

Proof. The conditions of Theorem **??** are fulfilled for:
T1) $n=3,\ x_1=t, x_2=u, x_3=v,\ C_1\in S/\lambda_x, C_2\in S/\lambda_y, C_3\in S/\rho_z$;
T2) $n=4, x_1=x, x_2=y, x_3=z, x_4=t, C_1\in S/\lambda_u, C_2\in S/\rho_v, C_3\in S/\lambda_u, C_4\in S/\rho_v$.
T3) $n=4, x_1=x, x_2=y, x_3=z, x_4=t, C_1\in S/\lambda_u, C_2\in S/\lambda_v, C_3\in S/\rho_w, C_4\in S/\rho_s$.
T4) $n=4, x_1=x, x_2=y, x_3=z, x_4=t, C_1\in S/\lambda_u, C_2\in S/\lambda_v, C_3\in S/\lambda_w, C_4\in S/\rho_s$. ■

Examples of such semigroups are the following bands ($S_1[7]$ and $S_2[7]$)

$$\begin{array}{cccc} 1 & 3 & 3 & 1 \\ 4 & 2 & 2 & 4 \\ 1 & 3 & 3 & 1 \\ 4 & 2 & 2 & 4 \end{array} \qquad \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 4 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4. \end{array}$$

They satisfy the condition T2) respectively for $x=u=v=1,\ y=4,\ z=2,\ t=3$ and $y=1,\ x=u=2,\ z=v=3,\ t=4$.

Theorem 2 *a) If the semigroup S has the property:*

$$\exists(x,y,z,t,u,v,w,)\in S^7: \quad \begin{array}{ccccccc} xu & \neq & xz & = & xt & \neq & xv \\ yv & \neq & yz & = & yu & \neq & yt \\ tw & \neq & zw & = & vw & \neq & uw \end{array}$$

then it is not orderable.

b)If the semigroup S has the property:

$$\exists(x,y,z,t,u,v,w)\in S^7: \quad \begin{array}{ccccccc} xu & \neq & xz & = & xt & \neq & xv \\ yv & \neq & yz & = & yu & \neq & yt \\ wt & \neq & wz & = & wv & \neq & wu \end{array}$$

then it is not left orderable.

Proof.a) We see that there exist three equivalence classes determined respectively by the inner translations $\lambda_x, \lambda_y, \rho_z$ which contain only one of the two-element subsets $\{z,t\}, \{z,u\}, \{z,v\}$ each. We denote them respectively by C_t, C_u, C_v and note that, by

Lemma 1, if the semigroup S is ordered by any stable order $<$, then they must be all convex. It is evident that in any linear order of S, at least two of the elements t, u, v must be at the same side of the element z. Let them consider, without any loss of generality, in the following position:

$$z < t < u.$$

The convexity of C_u yields $t \in C_u$ which contradicts the fact that $\{z, t\} \not\subset C_u$. So $<$ can not be left stable.
b) The proof in this case runs similarly to the previous one, the only difference is that all the considered inner translations are left ones. ■

An illustrative example of the dual of the last proposition is the semigroup $S_4[7]$:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4. \end{array}$$

Evidently, putting together (if possible) all the elements of a semigroup S that are in the same equivalence class determined by the left [right] inner translations of S does not automatically induce a left [right] stable order in it. The situation in any finite group, where the equivalence classes contained of one element, illustrates it. It also justifies the study in the following direction.
Let S be a semigroup which may have or not the identity element. We shall denote by S^1 the semigroup S with the adjoined identity, that is $S^1 = S \cup \{1\}$. With this notation we have:

Proposition 3 *If the semigroup S has the property*

$$\exists (x, y, z, t) \in (S^1)^4 : \qquad xz = yt \neq yz = xt \tag{3}$$

then it is not orderable. If the elements x, y, z, t of (??) are al different from the adjoined identity 1 of S^1, then S is neither left orderable nor right orderable.

Proof. We note that the condition (??) yields $z \neq t$ and $x \neq y$. First we shall prove the case when the elements x, y, x, t are all in S, that is, no one of them is the adjoined identity of the semigroup S. Let us suppose that there exists a left stable order $\leq$ in the semigroup S and that its elements z, t are such that $z < t$. Then

$$xz \leq xt \text{ and } yz \leq yt.$$

From the condition (??) the last inequalities are strict ones:

$$xz < xt, \quad yz < yt$$

which, also by the condition (??) means that they contradict each other.
Analogously one can prove that the relation $\leq$ cannot be right stable (if $x < y$ than $xz < yz$ and $xt < yt$).
For the other case, we note that no more than one of the elements in (??) can be the

adjoined identity (really, if $x = z = 1$, then $yt = 1$, and so on). So, if $x = 1$ than the obtained condition is

$$z = yt \neq yz = t$$

and if $z = 1$ it becomes

$$x = yt \neq xt = y.$$

In the first case S is not left orderable and in the second one it is not right orderable. The proof runs analogously to the previous one, where all the elements were in S, so it can be omitted. ■

Some of the semigroups that have the property (??) are the group $G = \{x, e\}$ of order 2 (for $z = t$ and $t = y = e$) and evidently any semigroup that has G as a subgroup. Other example is the following semigroup ([1], Appendix):

$$\begin{matrix} 3 & 4 & 3 & 6 & 1 & 6 \\ 1 & 5 & 3 & 4 & 2 & 6 \\ 3 & 6 & 3 & 6 & 3 & 6 \\ 3 & 1 & 3 & 6 & 4 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 3 & 3 & 6 & 6 & 6. \end{matrix}$$

The Condition (??) is fulfilled for $x = z = 2$, $y = t = 5$. It is seen that $xz = yt = 5 \neq 2 = xt = yt$.
We see that Condition (??) of Proposition ?? may be reformulated as:
$\exists(x, y, z, t) \in (S^1)^2$ such that xz, xt and yz, yt are different permutations of the same set $\{xz, yz\} = \{xt, yt\}$.
In these notations, Proposition ?? may be generalized as follows:

Theorem 3 *a)If S is a semigroup where $\exists k \in \aleph$, $\exists(x_1, x_2, \ldots, x_k, a, b) \in S^k \times (S^1)^2$ such that*

$$ax_1, ax_2, \ldots, ax_k \quad and \quad x_1 b, x_2 b, \ldots, x_k b$$

are two different permutations of the same set, then S is not orderable.
b)If S is a semigroup where $\exists k \in \aleph$, $\exists(x_1, x_2, \ldots, x_k, a, b) \in S^k \times (S^1)^2$ such that

$$ax_1, ax_2, \ldots, ax_k \quad and \quad bx_1, bx_2, \ldots, bx_k$$

are two different permutations of the same set, then S is not left orderable.

Proof.a) Let $\leq$ be a linear order in the set S and let us suppose the simplest case, when, according that order, the elements of the first permutation are in increasing order, that is

$$ax_1 < ax_2 < \cdots < ax_k \tag{4}$$

(If this is not the case, we may reorder simultaneously in the same way both the permutations and obtain the sequence (??)). It is clear that at most one of the elements a, b can be the adjoined identity ; the following proof is valid even for this case.
Let ax_i be the first of the elements of the first permutation such that $ax_i \neq x_i b$. Then it

is clear that $ax_i < x_ib$. Also, in the sequence (??) it must exist any $ax_p > ax_i$ such that $bx_p = ax_i$. Combining these inequalities we obtain

$$ax_p > ax_i \text{ and } x_pb < x_ib. \quad (5)$$

These inequalities mean that the order in S can not be stable. b) The proof in this case can be immediately obtained by the previous one substituting the elements $x_1b, x_2b, \ldots, x_kb$ respectively by the elements $bx_1, bx_2, \ldots, bx_k$. ■

The following semigroup of the fourth order (S 4; 24[6]) is clearly an example of such semigroups :

1	1	1	1
1	2	3	4
1	3	4	2
1	4	2	3

For $k = 2$ Theorem ??a) is more easily appliable in Proposition ??-like terms:

Corollary 3 *If the semigroup S has the property*

$$\exists(x, y, z, t) \in (S^1)^4 : \qquad xt = zy \neq zx = yt$$

then it is not linear (stable) orderable. ■

The following semigroup illustrates it. It is also one of the fourth order semigroups of the tables [6].

1	1	1	1
1	1	1	1
1	1	2	1
1	1	2	2

Evidently, we must take $x = z = 3, y = t = 4$ to conclude that any order in this semigroup can not be (two-sided) stable.

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