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ON PRIME k-TUPLES CONJECTURES

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Abstract

There exist cases when experiments with the computational part of the research could lead to new theoretical proofs. New hypotheses are suggested in concern with the Hardy–Littlewood conjecture that there exist infinitely many prime k-tuples. Although all the details on the presented investigation are given in another scope papers (applied mathematical logic), the formulation of the presented hypotheses has an independent role. Strong relations are revealed with the investigations from Number Theory period before the Hardy and Littlewood down to the greek system. The introduced formulas help to establish upper and lower bounds for **different** constellations of prime k-tuples.

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The most frequently used notions, signatures and short explanations

$[\alpha]$: the integer part of the number α. $[\alpha] \leq \alpha$.

Further the **natural numbers** are called all the nonnegative integers.

Z^+ : A set of positive integers.

a, b, k, n, x : Natural numbers.

d : Even numbers.

P : A set of prime numbers.

$p_n \in P$ or $p \in P$: A prime number, member of P.

x : A natural number (usually used as an argument for different functions). If x is given and no minimal interval value is mentioned, then (just for this paper) the function is considered on $[2, x]$.

$\pi(x)$: An amount of prime numbers $\leq x$.

$\pi_{b,a}(x)$: An amount of prime numbers in arithmetic progression $\{a + bn\}_{n=0}^{\infty}$ which are $\leq x$.

$P_x(p, p+d)$: An amount of couples of primes of the form p and $p+d$ which are $\leq x$.

$P_x(p, p+d_1, ..., p+d_{k-1})$: A number of those sequences $(a, a+d_1, ..., a+d_{k-1})$ for which $a+d_{k-1} \leq x$ and all the elements are primes with assumptions that d_i are even numbers (in ascending order); $d_1 \leq d_i \leq d_{k-1}$.

S_5 : A set of eight arithmetic progressions that contains all the primes ≥ 7 altogether with all the composites which divide them: $\{1+30n\}_{n=0}^{\infty}$, $\{7+30n\}_{n=0}^{\infty}$, $\{11+30n\}_{n=0}^{\infty}$, $\{13+30n\}_{n=0}^{\infty}$, $\{17+30n\}_{n=0}^{\infty}$, $\{19+30n\}_{n=0}^{\infty}$, $\{23+30n\}_{n=0}^{\infty}$, $\{29+30n\}_{n=0}^{\infty}$.

1 Introduction

Some cases could be quoted when elementary proof methods have a lot of advantages in comparison with more special, but sometimes more restricted in applications "higher techniques" methods. The reasons for that are hidden in the following.

- Less conditions are used in the proofs.
- Proof models are easily combined and verified.
- There exist more alternatives or ways to prove a chosen problem.
- Their application area is broader.
- Some of the reasons have concern with the **Occam Razor** usage

and so on. The most simple and clear-to-explain way is to state a result of a set of proofs as a hypothesis, to see the computational confirmations, and to trace its growth step by step to the desired conclusions. On the author's opinion this way sheds more light on useful information and/or connections with another problems than the isolated set of proofs.

There is no intention to explain almost everything in number theory with this kind of methods, but in two of the above mentioned cases such type of methods are welcome to work altogether with the other methods.

- When connections between different problems are traced.
- When the idea is considered that touches the core problems of Number Theory.

None of the definitions or hypotheses from the paper were obtained by analogy or with the usage of heuristic means. Only elementary number theory methods were used, touching the investigations on the formula $x/\log x$ [?, ?]. The next four sections have concern with the theoretical background. The explanations to the main formula are considered in section 6.

2 π_k- definitions

Definition 1.

Let k is a natural number and $\pi_0(x) = x$; $\pi_k(x)$ is defined by

$$\pi_k(x) = \sum_{p \leq x} \frac{\pi_{k-1}(p)}{p}, \quad x > 1 \tag{1}$$

Remarks to definition 1.

I. For $k = 1$ the formulation coincides with the "traditional case" but the interpretation in the middle of (2) is important:

$$\pi_1(x) = \pi(x) = \sum_{p \leq x} \frac{p}{p} = \sum_{p \leq x} 1; \tag{2}$$

II. If $[\pi_k(x)]$ $(k > 1)$ has relations with the prime k–tuple case $P_x(p, p + d_1, ...p + d_{k-1})$, then the π–signature could be more informative because it shows connections between different prime number constellations.

Otherwise the analysis of the newly defined $\pi_k(x)$ will shed light to different number theory problems.

III. The introduction of definition 1 will give stronger motivation for the usage in number theory of probabilistic, combinatorial, group, and mathematical logic methodologies. For example, there exists a connection between the cardinal numbers of the above sets and the probabilistic estimates.

Two hypotheses are presented below. Both of them are based on the usage of essential notions from definition 1 (sf. (??)).

3 Hypothesis

Hypothesis 1.

Let $P_x(p, p + d_1, ...p + d_{k-1})$ is considered and d_{k-1} is the maximal number from the ordered set $\{ d_1, d_2, ...d_{k-1} \}$ (in ascending order). Then

$$P_x(p, p + d_1, ...p + d_{k-1}) \underset{x > x_k^*}{>} \pi_k(x) \tag{3}$$

where $x_k^* < 10^4$ for $k = 2$, and for $k > 2$ it is less than for the case $k = 2$.

For bigger values of $d_{k-1} > 10^3$ additional (compensating) condition should be added to the corresponding $\pi_k(x)$ formula (??), e.g.:

$$\pi_2(x) = \sum_{p \leq x} \left(\frac{\pi(p + d_{k-1})}{p + d_{k-1}} - \frac{\pi(d_{k-1})}{d_{k-1}} \right) \tag{4}$$

4 Borders to P_x

For $k \geq 2$

$$\frac{x}{(\log x)^k} < \pi_k(x) < \frac{45}{32} 3.75^{k-2} \frac{(\pi(x))^k}{x^{k-1}}$$

The coefficients from the right hand side are obtained after the theoretical investigation of sieve method; they help for the establishment of corresponding $P_x(p, p + d_1, ...p + d_{k-1})$ borders.

Hypothesis 2.

For $k \geq 2$

$$\frac{x}{(\log x)^k} \underset{x \geq x^*_{k,1}}{<} P_x(p, p+d_1, ...p+d_{k-1}) \underset{x \geq x^*_{k,2}}{<} \frac{45}{16} 3.75^{k-2} \frac{(\pi(x))^k}{x^{k-1}} \tag{5}$$

where $x^*_{k,1}, x^*_{k,2}$ are the corresponding numbers, depending on k. [1]

5 Connections

Let the well known formula is considered (sf. fig. 1):

$$x \frac{1}{\log x} \underset{x \geq x^*}{>} \pi(x)$$

If the limit proof steps will have some relations to e.g. P. Chebyshev's proof way, then the next formula will not change the above inequality, and even make it stronger

$$x \frac{1}{\log x} \underset{x \geq x^*}{>} <\pi(x)\pi(x) \frac{\log x}{x}$$

More strong difference between the left and right parts is obtained in the following formula:

$$x \frac{1}{(\log x)^2} \underset{x \geq x^*}{>} \frac{(\pi(x))^2}{x}$$

On the k^{th} step the formula looks so:

$$x \frac{1}{(\log x)^k} < \frac{(\pi(x))^k}{x^{k-1}}$$

It is seen that almost all the differences at every step have influence on the left part of (??). For $k > 3$ the amounts for $x/(\log x)^k$ and $(\pi(x))^k$ diverge. The proof of the divergence is the base for the next results:

$$\lim_{x \to \infty} \frac{\pi_k(x)}{x/(\log x)^k} > 1$$

and

$$\lim_{x \to \infty} \frac{P_x(p, p+d_1 ..., p+d_{k-1})}{x/(\log x)^k} > 1$$

where k runs consequently all the natural numbers.

It is not difficult to establish new border formulas to P_x which are closer to the k-tuple amount than in the latter hypothesis, e.g.

[1] The right hand side inequality is valid only for the cases $d = 2p^s$, s is zero or natural number; $p = 2$ or $p \geq 7$. The other cases will be further considered.

$$P_x(p, p+d_1, ...p+d_{k-1}) \underset{x \geq x_k}{>} \frac{45}{32} 3.75^{k-1} \frac{(\pi(x))^{k+1}}{x^k} \quad (6)$$

where x_k depends on k. The problem is hidden in the fact that it is much harder to **prove** (not to calculate!) the limit theorems in the latter case (??).

6 Explanations

The coefficients in formula (??) were obtained from the following set of arithmetic progressions.

Definition 2.

S_5 is a set of eight arithmetic progressions that contains all the primes ≥ 7 altogether with all the composites which divide them: $\{1+30n\}_{n=0}^{\infty}$, $\{7+30n\}_{n=0}^{\infty}$, $\{11+30n\}_{n=0}^{\infty}$, $\{13+30n\}_{n=0}^{\infty}$, $\{17+30n\}_{n=0}^{\infty}$, $\{19+30n\}_{n=0}^{\infty}$, $\{23+30n\}_{n=0}^{\infty}$, $\{29+30n\}_{n=0}^{\infty}$.

The r.h.s. from formula (??) is represented in S_5 as (??).

$$P_x(p, p+d_1, ...p+d_{k-1}) < 2 \cdot \frac{N(\pi_{30,a}(x))^k}{(X_a)^{k-1}} \quad (7)$$

where a is one arbitrary chosen progression from S_5 (sf. definition 2); $\pi_{30,a}(x)$ is the number of primes in it $\leq x$; X_a is the number of the members of the progression $\leq x$; N is the number of the appearences of the considered constellation $P_x(p, p+d_1, ...p+d_{k-1})$ in S_5. E.g. $P_x(p, p+2)$ could appear only between 3 couples of progressions from S_5:

$\{11+30n\}_{n=0}^{\infty}$ and $\{13+30n\}_{n=0}^{\infty}$;
$\{17+30n\}_{n=0}^{\infty}$ and $\{19+30n\}_{n=0}^{\infty}$;
$\{29+30n\}_{n=0}^{\infty}$ and $\{31+30n\}_{n=0}^{\infty}$.

It should be mentioned that $N \leq 3$ for all the considered cases from formula (??).

It is known that $\pi_{30,a}(x) \sim \pi(x)/8$, $X_a \approx x/30$. For the constellations $P_x(p, p+2)$ quoted in formula (??) the coefficients are $(2 \cdot 3 \cdot 30)/64$, and the other coefficients are obtained in similar way.

The number 2 from formula (??) is introduced so that the r.h.s. be certainly bigger than $P_x(p, p+d_1, ...p+d_{k-1})$. As it follows from the previous Section, the l.h.s. of the formula (??) could be replaced with the asymptotically equivalent formulas:

$x/(\log x)^k \sim (\pi(x))^k/x^{k-1} \sim ...$

and the number of possible changes in the l.h.s. depends on n. Changed or not, the formula (??) contains $P_x(p, p+d_1, ...p+d_{k-1})$ altogether with all its oscillations. This data will be considered in further investigations.

7 Conclusions

Formulas are discussed in the paper concerning the infinity of k–tuples of primes. Definitions are given that are linked with prime k-tuples formulas. Hypotheses are considered on lower bounds for all the admissible prime constellations and on upper bounds for a big class of admissible prime constellations. The same way they are of use for inadmissible constellations research.

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