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SOME NEW EQUATIONS CONCERNING THE EULER FUNCTION

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Abstract This article presents some new equations concerning the Euler function. An equation for the sum $\sum_{d|n} d^a \cdot \varphi^b(d)$ is found by using the multiplicative property. This is applied to find the sums $\sum_{i=1}^{n}(i,n)^a$ and $\sum_{i=1}^{n} i \cdot (i,n)^a$.

1.Introduction

The Euler function is considered to be one of the most important function in Elementary Number Theory. This function has been intensely studied and many results have been proposed so far. These results concern applications of the function in Combinatorics, Algebra, Number Theory *etc.* In this condition, we may wonder if there still exists something new about the function to be proposed. This article proves that it may exist some aspects of the function that can be researched. In the following we will study the function in connection to the sums $\sum_{i=1}^{n}(i,n)^m$ and $\sum_{i=1}^{n} i \cdot (i,n)^m$.

A function $f : N \to N$ that satisfies $(n,m) = 1 \Rightarrow f(m \cdot n) = f(m) \cdot f(n)$ is called multiplicative function. It is known that if $f : N \to N$ is a multiplicative function, then $g : N \to N, g(n) = \sum_{d|n} f(n)$ is multiplicative too [1,4] . The most important example of multiplicative function is the Euler function defined by

$$\varphi : N^* \to N, \varphi(n) = \#\{i = \overline{1,n} : (i,n) = 1\}. \tag{1}$$

This is certainly one of the most studied function of Number Theory, many mathematicians have worked on it. Among the equations concerning φ proposed so far, the most simple and interesting is

$$\sum_{d|n} \varphi(d) = n \tag{2}$$

both from combinatorial and Number Theory point of view [1,4]. In the following, we will generalise Equation (2) by considering $\sum_{d|n} d^a \cdot \varphi^b(d)$, which is a connection between the function φ and the divisors of n.

we obtain from (*) the ordinary Fibonacci sequence: 0, 1, 1, 2,...

Therefore, the ordinary Fibonacci sequence can be represented by an $A-$progression. We shall show that some of the generalizations of this sequence can be represented by an A-progression, too. When a and b are fixed real numbers and f is a function defined by

$$f(1) = b - a, \ f(2) = b, \ f(k+2) = f(k+1) + f(k) + a,$$

we obtain from (*) the generalized Fibonacci sequence

$$a, b, \ a + b, \ a + 2.b, \ 2.a + 3.b, ...$$

(see e.g. [2]). When a, b and c are fixed real numbers and f is a function defined by

$$f(1) = b - a, \ f(2) = c - a, \ f(3) = b + c,$$

$$f(k+3) = f(k+2) + f(k+1) + f(k) + 2.a,$$

we obtain from (*) the generalized Fibonacci sequence named Tribonacci sequence (see e.g. [2]):

$$a, \ b, \ c, \ a + b + c, \ a + 2.b + 2.c, \ 2.a + 3.b + 4.c, ...$$

When a, b, c and d are fixed real numbers, and f and g are functions defined by:

$$f(1) = -a + b, \ f(2) = -a + c + d,$$

$$f(k+2) = g(k+1) + g(k) - a + 2.c \ (k \geq 1)$$

$$g(1) = -c + d, \ g(2) = a + b - c,$$

$$g(k+2) = f(k+1) + f(k) + 2.a - c \ (k \geq 1)$$

we obtain from (*) the generalization of the Fibonacci sequence from [3]. When for the same a, b, c and d

$$f(1) = -a + b, \ f(2) = -a + b + c,$$

$$f(k+2) = f(k+1) + g(k) + c \ (k \geq 1)$$

$$g(1) = -c + d, \ g(2) = a - c + d,$$

$$g(k+2) = g(k+1) + f(k) + a \ (k \geq 1)$$

we obtain from (*) the generalization of the Fibonacci sequence from [4-6].

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CONNECTIONS IN MATHEMATICS: FIBONACCI SEQUENCE VIA ARITHMETIC PROGRESSION

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The idea for this short remark was generated by the Marchisotto's paper [1]. Thus I borrow the first part of its title and offer to all colleagues to prepare a series of papers under the first part of this title.

Here we shall discuss an approach for an interpretation of the Fibonacci sequence as an arithmetic progression. The reasoning for this is the fact, that there is a relation between the way of generating the Fibonacci sequence and the way of generating the arithmetic progression. On the other hand, obviously, the Fibonacci sequence is not an ordinary arithmetic progression. Thus we can construct a new type of progression which will include both the ordinary arithmetic progression, and the Fibonacci sequences (the classical one and its generalizations).

Let $f : \mathcal{N} \rightarrow \mathcal{R}$ be a fixed function, where $\mathcal{N}$ and $\mathcal{R}$ are the sets of the natural and real numbers, respectively, and a be a fixed real number. The sequence

$$a,\ a+f(1),\ a+f(2),\ ...,\ a+f(k),\ ... \qquad (*)$$

we shall call A-progression (from "arithmetic progression").

Obviously, if $a_k = a + f(k)$ is its $k-$th member, then

$$\sum_{k=0}^{n} a_k = (n+1).a + \sum_{k=1}^{n} f(k).$$

When $f(k) = k.d$ for the fixed real number d we obtain from (*) the ordinary arithmetic progression.

When $a = 0$ and f is the function defined by:

$$f(1) = 1,\ f(2) = 1,\ f(k+2) = f(k+1) + f(k) \ \text{ for } k \geq 1,$$

for which a simple proof is presented in the following. Let $\{i_1, i_2, ..., i_{\varphi(n)}\}$ be the numbers that are relatively prime to n. Based on $\{i_1, i_2, ..., i_{\varphi(n)}\} = \{n - i_1, n - i_2, ..., n - i_{\varphi(n)}\}$, it follows that

$$2 \cdot \sum_{i=\overline{1,n},(i,n)=1} i = \sum_{i=\overline{1,n},(i,n)=1} i + \sum_{i=\overline{1,n},(i,n)=1} (n-i) = n \cdot \varphi(n).$$

Obviously, this equation does not hold for $n = 1$.

Proposition 3.3. If $m \neq 0$, then the equation

$$\sum_{i=1}^{n} i \cdot (i,n)^m = \frac{n^{m+1}}{2} \cdot f_{-m,1}(n) + \frac{n^{m+1}}{2}$$

holds.

Proof. Let $I_{n,d} = \{i = \overline{1,n} : (i,n) = d\}$ be the set of indices i that satisfies $(i,n) = d$. Obviously, this set satisfies

$$(\forall d|n)\, I_{n,d} = d \cdot I_{\frac{n}{d},1}. \tag{9}$$

The sum $\sum_{i=1}^{n} i \cdot (i,n)^m$ is transformed as follows:

$$\sum_{i=1}^{n} i \cdot (i,n)^m = \sum_{d|n} d^m \cdot \sum_{i \in I_{n,d}} i = \sum_{d|n} d^m \cdot d \cdot \sum_{i \in I_{\frac{n}{d},1}} i.$$

Equation (9) gives $\sum_{i \in I_{\frac{n}{d},1}} i = \frac{\frac{n}{d} \cdot \varphi\left(\frac{n}{d}\right)}{2}$ for each divisor $d \neq n$. Applying this, we find

$$\begin{aligned}
\sum_{i=1}^{n} i \cdot (i,n)^m &= n^{m+1} + \sum_{n \neq d|n} d^{m+1} \cdot \frac{\frac{n}{d} \cdot \varphi\left(\frac{n}{d}\right)}{2} = \\
&= n^{m+1} + \frac{n^{m+1}}{2} \cdot \sum_{n \neq d|n} \left(\frac{n}{d}\right)^{-m} \cdot \varphi\left(\frac{n}{d}\right).
\end{aligned}$$

Completing the last sum and changing the index sum, this becomes

$$\begin{aligned}
\sum_{i=1}^{n} i \cdot (i,n)^m = n^{m+1} - \frac{n^{m+1}}{2} + \frac{n^{m+1}}{2} \cdot \sum_{d|n} d^{-m} \cdot \varphi(d) = \\
= \frac{n^{m+1}}{2} + \frac{n^{m+1}}{2} \cdot f_{-m,1}(n).
\end{aligned}$$

Thus, the proposition holds.

From Theorem 2.2 and Proposition 3.3, we find the following result.

Theorem 3.4. If $n = p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots \cdot p_s^{k_s}$ is the prime number decomposition, then the following equation holds

$$\sum_{i=1}^{n} i \cdot (i,n)^m = \frac{n^{m+1}}{2} + \frac{n^{m+1}}{2} \cdot \begin{cases} \prod_{i=1}^{s} \left[1 + k_i \cdot \left(1 - \frac{1}{p_i}\right)\right], m = 1 \\ \prod_{i=1}^{s} \left[1 + \left(1 - \frac{1}{p_i}\right) \cdot \frac{p_i^{(1+m)\cdot(1-k_i)} - 1}{p_i^{1-m} - 1}\right], m \neq 1 \end{cases} \tag{10}$$

for any number $m \neq 0$.

Equation (10) gives the following particular equations:

- $m = 1 \Rightarrow \sum_{i=1}^{n} i \cdot (i,n) = \frac{n^2}{2} + \frac{n^2}{2} \cdot \prod_{i=1}^{s} \left[1 + k_i \cdot \left(1 - \frac{1}{p_i}\right)\right].$
- $m = -1 \Rightarrow \sum_{i=1}^{n} \frac{i}{(i,n)} = \frac{1}{2} + \frac{1}{2} \cdot \prod_{i=1}^{s} \frac{p_i^{2\cdot k_i+1}+1}{p_i+1}.$
- $m = 2 \Rightarrow \sum_{i=1}^{n} i \cdot (i,n)^2 = \frac{n^3}{2} + \frac{n^3}{2} \cdot \prod_{i=1}^{s} \left(1 + \frac{1}{p_i} - \frac{1}{p_i^{k_i+1}}\right).$

Equations (8) and (10) gives the following interesting consequence

$$\sum_{i=1}^{n} i \cdot (i,n)^m = \frac{n^{m+1}}{2} + \frac{n}{2} \cdot \sum_{i=1}^{n} (i,n)^m \tag{11}$$

For $m = 1$ this implies $\sum_{i=1}^{n} \frac{i}{(i,n)} = \frac{1}{2} + \frac{1}{2} \cdot \sum_{i=1}^{n} \frac{n}{(i,n)}$ that represents the equation between the Tabirca functions [3,5].

In spite of the fact that the Euler function has been studied for a very long time, this article has proved that it is still possible to find new facts about it. After a detailed and careful investigation, we have found that Equations (5), (8) and (10) have not been probably proposed until now. Therefore, we can consider that they are new equations concerning the Euler function.

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