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SOME REMARKS CONCERNING THE BERNOULLI NUMBERS

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Abstract The aim of this article is to propose some remarks on the Bernoulli numbers. Firstly, a simple proof for the the equation $B_{2n+1} = 0$ is presented. This proof also gives an equation for $\zeta(2k)$. Using a simple computation, the values of $\zeta(2k), k = 1, 12$ are presented. Finally, an equation for the infinite product $\prod_p \frac{p^{2k}-1}{p^{2k}+1}$ is proposed based on the Bernoulli numbers.

Introduction

The Bernoulli numbers $(B_n)_{n \geq 0}$are important combinatorial numbers with many application in Number Theory. Among several definitions of these numbers have been proposed so far (see [2], [4], [5], [7]), we prefer the recursive one. The sequence of the Bernoulli numbers $(B_n)_{n \geq 0}$ is recursively defined by

$$B_0 = 1 \tag{1}$$

$$B_n = \frac{-1}{n+1} \cdot \left(\sum_{i=0}^{n-1} C_{n+1}^i \cdot B_i \right). \tag{2}$$

The main advantage of this definition is computability, which means that B_n can be easier found from $B_0, B_1, ..., B_{n-1}$. Using these above equations and a simple computation, we can easier find these number (see Table 1).

n	B_n	n	B_n	n	B_n	n	B_n	n	B_n
0	1	5	0	10	$\frac{5}{66}$	15	0	20	$-\frac{174611}{330}$
1	$-\frac{1}{2}$	6	$\frac{1}{42}$	11	0	16	$-\frac{3617}{510}$	21	0
2	$\frac{1}{6}$	7	0	12	$-\frac{691}{2730}$	17	0	22	$\frac{854413}{138}$
3	0	8	$-\frac{1}{30}$	13	0	18	$\frac{43867}{798}$	23	0
4	$-\frac{1}{30}$	9	0	14	$\frac{7}{6}$	19	0	24	$-\frac{236364091}{2730}$

Table 1. The Bernoulli numbers

There are several important results in Number Theory that involve the Bernoulli numbers. An interesting application to the computation of the sum $\sum_{i=1}^{n} i^k$ was recently proposed by Bencze [2]. This results gives the equation

$$\sum_{i=1}^{n} i^k = \frac{1}{k+1} \cdot n^{k+1} + \frac{1}{2} \cdot C_{k+1}^1 \cdot n^k + C_{k+1}^2 \cdot B_2 \cdot n^{k-1} + ... + C_{k+1}^k \cdot B_n \cdot n. \tag{3}$$

What we want to point is that the Bernoulli numbers can be applied to find an equation for something quite similar, the series $\zeta(2k) = \sum_{n>0} \frac{1}{n^{2\cdot k}}$. Although this is an classical result, we present a simplified proof for it. From this proof, we can obtain that $B_{2n+1} = 0$.

The Main Result

The harmonic series $\sum_{n>0} \frac{1}{n^a}$ is one of the most studied series, many results have been proposed so far. This series is convergent for $a \geq 1$ and its sum is given by the Riemann function

$$\sum_{n>0} \frac{1}{n^a} = \zeta(a).$$

For many reasons, it would be useful, to know the $\zeta(a)$ for all integers $a \geq 2$. These values are known for even numbers, but unfortunately are not known for odd numbers. The only important result concerning $\zeta(a)$ with a odd is that $\zeta(3)$ is irrational. This was a quite long-standing conjecture that was proved by Apery in 1979 (according to Jones [5]). In order to obtain the equation for $\zeta(2k)$, we use an simple way that was used by Apostol [1] for $\zeta(2)$. A similar proof can be found in Jones [5].

We start from the infinite product expansion of $\sin z$

$$\sin z = z \cdot \prod_{n>0} \left(1 - \frac{z^2}{n^2 \cdot \pi^2}\right)$$

which gives by taking logarithms

$$\ln \sin z = \ln z + \sum_{n>0} \ln \left(1 - \frac{z^2}{n^2 \cdot \pi^2}\right).$$

We have by differentiating the last equation term by term the following equation

$$\cot z = \frac{1}{z} - \sum_{n>0} \frac{2 \cdot z}{n^2 \cdot \pi^2} \cdot \left(1 - \frac{z^2}{n^2 \cdot \pi^2}\right)^{-1}. \tag{4}$$

Now, we use the geometric series that gives

$$\left(1 - \frac{z^2}{n^2 \cdot \pi^2}\right)^{-1} = \sum_{k\geq 0} \left(\frac{z^2}{n^2 \cdot \pi^2}\right)^k = \sum_{k\geq 0} \frac{z^{2k}}{n^{2k} \cdot \pi^{2k}}$$

and collect the power of z in Equation (4) obtaining

$$\cot z = \frac{1}{z} - 2\sum_{n>0}\sum_{k\geq 0} \frac{z^{2k+1}}{n^{2k+2} \cdot \pi^{2k+2}} = \sum_{k>0} \frac{z^{2k-1}}{\pi^{2k}} \sum_{n>0} \frac{1}{n^{2k}} = \sum_{k>0} \frac{z^{2k-1}}{\pi^{2k}} \zeta(2k). \tag{5}$$

This represents one final point in the expansion of $\cot z$.

Now, we use the series expansion that gives the Bernoulli numbers [5], [7]

$$\frac{t}{e^t - 1} = \sum_{m\geq 0} \frac{B_m}{m!} \cdot t^m. \tag{6}$$

The expression $\frac{t}{e^t-1}$ is transformed as follows:

$$\begin{aligned}\frac{t}{e^t-1} &= \frac{t}{2}\cdot\left(\frac{e^t+1}{e^t-1}-1\right) = \\ \frac{t}{2}\cdot\left(\frac{e^{t/2}+1}{e^{t/2}-1}-1\right) &= \frac{t}{2}\cdot\left(\coth\frac{t}{2}-1\right).\end{aligned}$$

Putting $z=\frac{i\cdot t}{2}$ and based on $i\cdot\cot i\cdot x=\coth x$, we get

$$\frac{t}{e^t-1}=\frac{t}{2}\cdot\left(i\cot\frac{i\cdot t}{2}-1\right)=z\cdot\cot z+i\cdot z.$$

This gives another final point of the expansion of $\cot z$

$$\cot z=-i+\frac{1}{z}\sum_{m\geq 0}\frac{B_m}{m!}\cdot\left(\frac{2z}{i}\right)^m=-i+\sum_{m\geq 0}\frac{B_m}{m!}\cdot\left(\frac{2}{i}\right)^m\cdot z^{m-1}. \quad (7)$$

Comparing the expansions from Equations (5) and (7), we find two important cases.

Case 1. If $m=2k-1>1$ then $\frac{B_m}{m!}\cdot\left(\frac{2}{i}\right)^m\cdot z^{m-1}=0$ that gives $B_{2k-1}=0$. Moreover, this result could be anticipated from Table 1.

Case 2. If$m=2k>1$ then

$$\frac{-2}{\pi^{2k}}\cdot\zeta(2k)=\frac{B_{2k}}{(2k)!}\cdot\left(\frac{2}{i}\right)^{2k}$$

so that

$$\zeta(2k)=\frac{B_{2k}}{(2k)!}\cdot(-1)^{k-1}\cdot 2^{2k-1}\cdot\pi^{2k}.$$

Thus, we have obtained two important results, which are presented by the following theorems.

Theorem 1. If $n>1$ is an odd integer number, then $B_n=0$.

Theorem 2. The equation of the Riemann function for the even integer numbers $2k>1$ is

$$\zeta(2k)=\frac{B_{2k}}{(2k)!}\cdot(-1)^{k-1}\cdot 2^{2k-1}\cdot\pi^{2k}. \quad (8)$$

Theorem 2 gives the equations for the first harmonic series with the odd integer power. Using a simple computation, we have obtained the following equations.

$$\sum_{n>0}\frac{1}{n^2}=\zeta(2)=B_2\cdot\pi^2=\frac{\pi^2}{6}$$

$$\sum_{n>0}\frac{1}{n^4}=\zeta(4)=-\frac{B_4\cdot\pi^4}{3}=\frac{\pi^4}{90}$$

$$\sum_{n>0}\frac{1}{n^6}=\zeta(6)=\frac{2\cdot B_6\cdot\pi^6}{45}=\frac{\pi^6}{945}$$

$$\sum_{n>0}\frac{1}{n^8}=\zeta(8)=-\frac{B_8\cdot\pi^8}{105}=\frac{\pi^8}{9450}$$

$$\sum_{n>0}\frac{1}{n^{10}}=\zeta(10)=\frac{2\cdot B_{10}\cdot\pi^{10}}{14175}=\frac{\pi^{10}}{93555}$$

$$\sum_{n>0}\frac{1}{n^{12}}=\zeta(12)=-\frac{2\cdot B_{12}\cdot\pi^{12}}{467775}=\frac{691}{638512875}\cdot\pi^{12}$$

$$\sum_{n>0}\frac{1}{n^{14}}=\zeta(14)=\frac{4\cdot B_{14}\cdot\pi^{14}}{42567525}=\frac{2}{18243225}\cdot\pi^{14}$$

$$\sum_{n>0}\frac{1}{n^{16}}=\zeta(16)=-\frac{2\cdot B_{16}\cdot\pi^{16}}{638512875}=\frac{3617}{32\,56415\,66250}\cdot\pi^{16}$$

$$\sum_{n>0}\frac{1}{n^{18}}=\zeta(18)=\frac{2\cdot B_{18}\cdot\pi^{18}}{97692469875}=\frac{43867}{3897\,92954\,80125}\cdot\pi^{18}$$

$$\sum_{n>0}\frac{1}{n^{20}}=\zeta(20)=-\frac{2\cdot B_{20}\cdot\pi^{20}}{928\,07846\,38125}=\frac{174611}{1\,53132\,94652\,90625}\cdot\pi^{20}$$

$$\sum_{n>0}\frac{1}{n^{22}}=\zeta(22)=\frac{4\cdot B_{22}\cdot\pi^{22}}{2\,14386\,12514\,06875}=\frac{4982}{43127\,23800\,20625}\cdot\pi^{22}$$

$$\sum_{n>0}\frac{1}{n^{24}}=\zeta(24)=-\frac{2\cdot B_{24}\cdot\pi^{24}}{147\,92642\,63470\,74375}=\frac{18909\,12728}{67306\,52398\,79188\,40625}\cdot\pi^{24}$$

The Computation of the Product $\prod_p \frac{p^{2k}-1}{p^{2k}+1}$

An interesting application of the Bernoulli numbers is represented by the computation of the product $\prod_p \frac{p^{2k}-1}{p^{2k}+1}$. This starts from the well-known equation [4], [5], [6]

$$\prod_p\frac{1}{1-p^{-a}}=\sum_{n>0}\frac{1}{n^a}=\zeta(a) \qquad (9)$$

which connects the product over the set of prime numbers to the harmonic series. There are several interesting problem concerning products over primes [3]. One of these asks to prove that the product $\prod_p \frac{p^{2k}-1}{p^{2k}+1}$ is rational. In the following, we propose an equation for proving that.

Theorem 3. If $k>1$ is an integer number, then

$$\prod_p\frac{p^{2k}-1}{p^{2k}+1}=-2\cdot\frac{B_{4k}}{(B_{2k})^2}\cdot\frac{1}{C_{4k}^{2k}}. \qquad (10)$$

The proof is based on Equation (9), which gives

$$\prod_p\frac{p^{2k}-1}{p^{2k}+1}=\prod_p\frac{\left(p^{2k}-1\right)^2}{p^{4k}-1}=\prod_p\frac{\left(1-\frac{1}{p^{2k}}\right)^2}{1-\frac{1}{p^{4k}}}=\frac{\zeta(4k)}{\zeta^2(2k)}.$$

Putting in the last equation, the values of the Riemann function, we find

$$\prod_p\frac{p^{2k}-1}{p^{2k}+1}=\frac{\zeta(4k)}{\zeta^2(2k)}=\frac{\frac{B_{4k}}{(4k)!}\cdot(-1)^{2k-1}\cdot 2^{4k-1}\cdot\pi^{4k}}{\left(\frac{B_{2k}}{(2k)!}\cdot(-1)^{k-1}\cdot 2^{2k-1}\cdot\pi^{2k}\right)^2}=$$

$$= -2 \cdot \frac{B_{4k}}{(B_{2k})^2} \cdot \frac{((2k)!)^2}{(4k)!} = -2 \cdot \frac{B_{4k}}{(B_{2k})^2} \cdot \frac{1}{C_{4k}^{2k}}.$$

Obviously, the product $\prod_p \frac{p^{2k}-1}{p^{2k}+1} = -2 \cdot \frac{B_{4k}}{(B_{2k})^2} \cdot \frac{1}{C_{4k}^{2k}}$ is rational.

Finally, we present some particular equations of (10).

$$\prod_p \frac{p^2-1}{p^2+1} = -2 \cdot \frac{B_4}{(B_2)^2} \cdot \frac{1}{C_4^2} = \frac{2}{5}$$

$$\prod_p \frac{p^4-1}{p^4+1} = -2 \cdot \frac{B_8}{(B_4)^2} \cdot \frac{1}{C_8^4} = \frac{6}{7}$$

$$\prod_p \frac{p^8-1}{p^8+1} = -2 \cdot \frac{B_{16}}{(B_8)^2} \cdot \frac{1}{C_{16}^8} = \frac{7234}{7293}$$

Final Remarks

The Bernoulli numbers have proved to be important both in Combinatorics and Number Theory. The article has proposed a simple proof for the fact that the Bernoulli numbers of odd index are zero. This article has also given a simple proof for Equation (7) and presented some particular equations for the harmonic series.

From space limitation reason, we have shown only the values of the Bernoulli numbers for small values. Based on Equations (1)-(2), we can extend the computation for large values of n.

References

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