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SOME ASPECTS OF THE DOMINANT ROOT OF A CHARACTERISTIC POLYNOMIAL

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Abstract

This paper considers some properties associated with the dominant root of the characteristic polynomial of arbitrary order linear homogeneous recurrence relations with integer coefficients. In particular, it looks at computational issues for the general terms of the related sequences, and gives examples in terms of the Fibonacci numbers.

1. Introduction

Define a fundamental homogeneous linear recursive sequence of arbitrary order r by the recurrence relation

$$u_n = \sum_{j=1}^{r} P_j u_{n-j}, \quad n > 0,$$

with $u_n = 0, n < 0$, and $u_0 = 1$, as initial conditions. The coefficients are arbitrary numbers: integers if we want integer sequences. We shall be concerned in this note with some aspects of the dominant root α of the auxiliary equation of the above recurrence relation.

The fundamental nature of $\{u_n\}$ has been illustrated by d'Ocagne (Dickson, 1952) who showed that any element of the set $\Omega = \Omega(P_1, P_2, ..., P_r)$ of all sequences of order r which satisfy the recurrence relation can be expressed in terms of the fundamental sequence and the initial terms (of $\{w_n\}$ say):

$$w_n = \sum_{j=0}^{r-1} \sum_{k=j}^{r-1} (-1)^{k-j} P_{k-j} w_j u_{n-k}, \quad n \geq 0,$$

with $P_0 = 1$ for notational convenience. For example, when $r = 2$, we get

$$w_n = w_1 u_{n-1} + w_0(u_n - P_1 u_{n-1}) = w_1 u_{n-1} + P_2 w_0 u_{n-2}$$

which agrees with Equation (3.14) of Horadam (1965).

2. Fibonacci Results

Theorem 1

$$F_{n+1} = [\alpha F_n]$$

Corollary 1

$$\text{An integer } m = [F_{n+1}/\alpha] \text{ iff } m = F_n$$

where [•] is the nearest integer function (Gilman and Rose, 1984) and α is the dominant root of the characteristic equation associated with the Fibonacci recurrence relation.

For example, the theorem is a generalization of the result of Hoggatt and Bicknell (1979) that for $n > 2$

$$F_{n+1} = \left\{ \begin{array}{ll} \lfloor \alpha F_n \rfloor, & n \text{ odd}, \\ \lfloor \alpha F_n \rfloor + 1, & n \text{ even}, \end{array} \right\}$$

in which $\lfloor \bullet \rfloor$, the floor function, is the integer part of a number.

$$\begin{aligned} F_{n+1} &= \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \\ &= (\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}) - \beta^n(\frac{\alpha - \beta}{\alpha - \beta}) + \beta^n \\ &= \frac{\alpha^{n+1} - \alpha\beta^n}{\alpha - \beta} + \beta^n \\ &= \alpha F_n + \beta^n \quad (\text{and } |\beta^n| < 0.5, \ n > 1). \blacksquare \end{aligned}$$

For example,

$$\begin{aligned} F_8 &= \lfloor 1.61803 \times 13 \rfloor \\ &= \lfloor 21.03439 \rfloor \\ &= 21; \\ F_7 + F_6 &= \lfloor 1.61803 \times 8 \rfloor + \lfloor 1.61803 \times 5 \rfloor \\ &= \lfloor 12.94424 \rfloor + \lfloor 8.09018 \rfloor \\ &= 13 + 8. \end{aligned}$$

The corollary is amenable to computational testing, namely, given an integer m to tes whether $m \in \{F_n\}$, that is whether it is a Fibonacci number. For instance, consider $m_1 = 12586269025$, then $m_1/\alpha = 7778742049$ is "extremely close" to an integer, (and $m_1 = F_{50}$), whereas if $m_2 = 12586269026$ ($= m_1 + 1$), then m_2/α is not "very close" to an integer (and $m_2 \notin \{F_n\}$). More specifically, $[F_{50}/\alpha] = F_{49} = m_1$ and $[\alpha \times m_1] = F_{50}$.

It is the purpose of this paper to explore some aspects of these phenomena, though the nature of the approximations leaves some of the questions unresolved.

3. Contraction Process

If the conjectures are true, then it is important to find a computationally efficient method of calculating dominant roots for $r > 2$. We shall develop a contraction process for arbitrary $r > 2$ after first illustrating the process with a third-order case. Consider $r = 3$, $P_1 = 0$, $P_2 = P_3 = 1$, so that

$$w_n = w_{n-2} + w_{n-3}.$$

The characteristic equation is then

$$x^3 = x + 1.$$

so that, in turn,

$$\begin{aligned} x^4 &= x^2 + x \\ x^5 &= x^3 + x^2 = x^2 + x + 1 \\ x^6 &= x^3 + x^2 + x = x^2 + 2x + 1. \end{aligned}$$

In order to speed up the process, Gnanadoss (1960) suggested a contraction as follows.

$$x^6 = (x^3)^2 = x^2 + 2x + 1$$

and so on until

$$x^{48} = 170625x^2 + 2260302x + 128801,$$

and

$$x^{49} = 226030x^2 + 299426x + 170625,$$

so that the dominant root of the characteristic equation is

$$\alpha = x^{49}/x^{48}$$

$$= \left\{ \begin{array}{ll} 1.324717957 & \text{when } x = 1, \\ 1.324717955 & \text{when } x = 2, \\ 1.324717973 & \text{when } x = 1.3. \end{array} \right\}$$

If we take the last value of α then Theorem 1 works for

$$u_{n+3} = u_{n+1} + u_n.$$

For the case of arbitrary order r, the characteristic equation can be rewritten similarly as

$$\begin{aligned}
x^r &= \sum_{j=1}^{r} P_j x^{r-j} \\
x^{r+1} &= \sum_{j=1}^{r} P_j x^{r-j+1} \\
&= P_1 x^r + \sum_{j=2}^{r} P_j x^{r-j+1} \\
&= \sum_{j=1}^{r} P_1 P_j x^{r-j} + \sum_{j=1}^{r-1} P_{j+1} x^{r-j} \\
&= \sum_{j=1}^{r} (P_1 P_j + P_{j+1}) x^{r-j} \\
x^{r+2} &= \sum_{j=1}^{r} (P_1 P_j + P_{j+1}) x^{r-j+1} \\
&= (P_1^2 + P_2) x^r + \sum_{j=2}^{r} (P_1 P_j + P_{j+1}) x^{r-j+1} \\
&= \sum_{j=1}^{r} (P_1^2 + P_2) P_j x^{r-j} + \sum_{j=1}^{r} (P_1 P_{j+1} + P_{j+2}) x^{r-j} \\
&= \sum_{j=1}^{r} (P_1^2 P_j + P_2 P_j + P_1 P_{j+1} + P_{j+2}) x^{r-j}.
\end{aligned}$$

More generally,

Theorem 2

$$\text{if } x^r = \sum_{j=0}^{r-1} P_{r-j} x^j, \text{then } x^{r+n} = \sum_{j=0}^{r-1} Q_{n,j} x^j,$$

where

$$\begin{aligned}
Q_{n,0} &= P_r u_n \\
Q_{0,m} &= P_{r-m} \\
Q_{n,m} &= 0 \text{ for } m \geq r, m < 0, n < 0 \\
Q_{n,m} &= Q_{n-1,m-1} + P_{r-m} Q_{n-1,r-1}, 0 < m < r.
\end{aligned}$$

Proof We use induction on n.

$$\begin{aligned}
\sum_{j=0}^{r-1} Q_{0,j} x^j &= \sum_{j=0}^{r-1} P_{r-j} x^j \\
&= x^{r+0}.
\end{aligned}$$

Assume the result is true for $n = 1,2,3,\ldots,s$.

$$
\begin{aligned}
x^{r+s+1} &= \sum_{j=0}^{r-1} Q_{s,j}x^j \\
&= Q_{s,r-1}x^r + \sum_{j=0}^{r-2} Q_{s,j}x^{j+1} \\
&= Q_{s,r-1}x^r + \sum_{j=0}^{r-1} Q_{s,j-1}x^j \\
&= Q_{s,r-1}x^r + \sum_{j=0}^{r-1} Q_{s+1,j}x^j - \sum_{j=0}^{r-1} P_{r-j}Q_{s,r-1}x^j \\
&= Q_{s,r-1}x^r - Q_{s,r-1}\sum_{j=0}^{r-1} P_{r-j}x^j + \sum_{j=0}^{r-s} Q_{s+1,j}x^j \\
&= \sum_{j=o}^{r-1} Q_{s+1,j}x^j. \qquad \blacksquare
\end{aligned}
$$

In the case when $r = 2$, the theorem reduces to the known result that $\alpha \to F_{n+1}/F_n$ as $n \to \infty$. From the study of the convergents p_n, q_n of the continued fraction expansion of $\alpha = [1; \dot{1}]$ the error is less than $1/q_n^2$ (Mack, 1970a,b). For instance,

$$\frac{F_{20}}{F_{19}} = \frac{6765}{4181} = 1.61803396317$$

so that the Fibonacci auxiliary polynomial $\alpha^2 - \alpha - 1$ is satisfied to the ninth decimal place for n as low as 20.
At this stage one might note that Goldstern et al (1989) have determined the asymptotic distribution function of the ratios of the terms of a linear recurrence. In doing so they too have studied the characteristic polynomials. de Pillis (1998) has highlighted fascinating and surprising feature's of Newton's formula for finding a root of a non-linear function when applied to cubic polynomials and has speculated on the generalization of his observations.

4. Some Examples

Table 1 shows the first few terms of $\{Q_{n,m}\}$ with $P_i = 1$.

r	2		3			4				5				
$n\backslash m$	0	1	0	1	2	0	1	2	3	0	1	2	3	4
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	2	1	2	1	1	2	2	1	1	2	2	2	1
2	2	3	1	2	3	1	2	3	1	1	2	3	3	1
3	3	5	3	4	5	1	2	3	4	1	2	3	4	1
4	5	8	5	8	9	4	5	6	7	1	2	3	4	5
5	8	13	9	14	17	7	11	12	13	5	6	7	8	9
6	13	21	17	26	31	13	20	24	25	9	14	15	16	17
7	21	34	31	48	57	25	38	45	49	17	26	31	32	33
8	34	55	57	88	105	49	74	87	94	33	50	59	64	65
9	55	89	105	162	193	94	143	168	181	65	98	115	124	129

Table 1: $Q_{n,m}, r = 2,3,4,5$

Consider further that for $\{u_n\}, P_j = 1, j1, 2, \ldots, r$,
when $r = 1$,

$$u_n = u_{n-1} \text{ and } \alpha = 1;$$

when $r = 2$,

$$u_n = u_{n-1} + u_{n-2} \text{ and } \alpha = 1.62.$$

In general,

Corollary 2

$$\lim_{r \to \infty} \alpha(r) = 2.$$

Proof $x^r = \sum_{j=1}^{r} x^{r-j} = \frac{x^r - 1}{x - 1}$ implies $\alpha = 2 - \frac{1}{\alpha^r}$ (see Table 2). ■

r	α
1	1
2	1.618033989
3	1.839286755
4	1.927561975
5	1.968948237
6	1.983582843
7	1.991964197
8	1.99603118
9	1.99802947

Table 2: Dominant Root Values

Note that if we use the value of α from Table 2 with the recursive sequence $\{u_n\}$ defined by

$$u_{n+3} = u_{n+2} + u_{n+1} + u_n,$$

then Theorem 1 does not work for this sequence.

5. Further Connections

Observe that the characteristic equation, $x^3 - x - 1$, is of interest because it represents a special case of polynomials of degree > 2, in so far as its only real root ω turns out to be the fundametnal unit of $Q(\omega)$. As we saw, $\omega = 1.325$, with conjugate zeros $\omega' = -0.662 + 0.562i$, $\omega" = -0.662 - 0.562i$ and $|\omega'| = |\omega"| = 0.868$. Bernstein (1874) has proved that $(1, \omega, \omega^2)$ is a minimal basis of $Q(\omega)$. He used a recursive approach to establish in turn that

$$\omega^n = r_n + s_n\omega + t_n\omega^2$$

and so

$$\begin{aligned} \omega^{n+1} &= r_n\omega + s_n\omega^2 + t_n(1+\omega) \\ &= r_{n+1} + s_{n+1}\omega + t_{n+1}\omega^2, \end{aligned}$$

and hence, by comparing coefficients,

$$\begin{aligned} r_{n+1} &= 0r_n + 0s_n + 1t, \\ s_{n+1} &= 1r_n + 0s_n + 1t_n, \\ t_{n+1} &= 0r_n + 1s_n + 0t'_n \end{aligned}$$

and

$$\begin{aligned} s_{n+1} &= r_n + r_{n+1}, \\ t_{n+1} &= r_{n-1} + r_n. \end{aligned}$$

Let

$$R(x) = \sum_{n=0}^{\infty} r_n x^n.$$

Then using

$$T_{n+3} = T_n + T_{n+1},$$

we get

$$R(x) = \frac{1-x^2}{1-x^2-x^3}$$

and

$$r_n = \sum_{k=0}^{\infty} \binom{m-k-1}{2k-1+2\mu}$$

in which $m = \lfloor \frac{1}{2}n \rfloor$, and $\mu = \frac{1}{2}n - m$. Thus $x^3 - x - 1$ is the recursion function for

$$f(n,2) = \sum_{j=0} (-1)^j \binom{n-2j}{j}.$$

Thus Bernstein (1974) has shown, with methods similar to those considered here, but from a different point of view, that the question of the zeros of $f(n,2)$ is a combinatorial one. He observed further that the study of $f(n,2)$ for real values of n and of

$$f(n,k) = \sum_{j=0} (-1)^j \binom{n-kj}{j}$$

is an open one. Moreover, the conjugate function

$$\begin{aligned} g(n) &= (-1)^n f(n,2) \\ &= r_{n+3}^2 - r_{n+2}r_{n+4} \end{aligned}$$

takes the values of all perfect squares from 1 to 49, (that is, 1,4,9,16,25,36,49) between $n = 1$ and $n = 28$, but it is not known if it takes on all perfect squares as values. The case $f(n,1)$ yields the Fibonacci numbers, as is well known, and they too are amenable to a combinatorial explanation (Hoggatt and Lind, 1968).

More recently, Rieger (1999) has applied Newton approximation to the Golden Section (effectively the dominant root of the second order case). His consideration of the continued fraction convergents in this context has been developed by Moore (1993) who has also considered the asymptotic behaviour of golden numbers (Moore, 1994), as has Prodinger (1996).

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