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ONE EXTREMAL PROBLEM. 9

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The n-th member of the sequence of the r-angle number is obtained by the formula $(r \geq 3, n \geq 1)$:

$$P_n^r = \frac{n((r-2)n-r+4)}{2}.$$

Let $n \geq 4$ be a fixed natural number. Therefore, for the natural number $k \geq 1$:

$$P_k^{n-k} = \frac{k((n-k-2)k-n+k+4)}{2}.$$

Let us consider

$$m_k \equiv 2.P_k^{n-k} = k((n-k-2)k-n+k+4),$$

i.e.,

$$m_k = k^2n - k^3 - k^2 - kn + 4k. \tag{1}$$

We shall study the properties of the sequence $\{m_k\}_{k=1}^{n-3}$.

Let us construct the system

$$\begin{cases} m_k < m_{k-1} \\ m_k < m_{k+1} \end{cases} \tag{2}$$

From (1) it follows, that the system (2) is equivalent to the system

$$\begin{cases} -3k^2 + 2kn - 5k + 2 > 0 \\ 3k^2 - 2kn + 2n - k + 4 > 0 \end{cases} \tag{3}$$

The solutions of the first equality of (3) are elements of the set

$$\{x \mid 0 \leq x < \frac{2n-5+\sqrt{4n^2-20n+49}}{6}\} \tag{4}$$

(because $2n - 5 < \sqrt{4n^2-20n+49}$), while the solutions of the second equality of (3) are elements of the set

$$\{x \mid \frac{2n+1+\sqrt{4n^2-20n+49}}{6} \leq x < +\infty) \tag{5}$$

(because the set $\{x \mid -\infty \leq x < \frac{2n+1-\sqrt{4n^2-20n+49}}{6})\}$ does not contain natural numbers, because $\frac{2n+1-\sqrt{4n^2-20n+49}}{6} < 1$, as we mentioned above).

From (4) and (5) it is seen directly, that there are no natural numbers, which are solutions of the system (3), and therefore, the sequence $\{m_k\}_{k=1}^{n-3}$ does not have an internal minimal element.

Now, we construct the system

$$\begin{cases} m_k \geq m_{k-1} \\ m_k \geq m_{k+1} \end{cases} \tag{6}$$

From (1) it follows, that the system (6) is equivalent to the system

$$\begin{cases} 3k^2 - 2kn + 5k - 2 \geq 0 \\ 3k^2 - 2kn + 2n - k - 4 \leq 0 \end{cases} \tag{7}$$

The solutions of the first equality of (7) are elements of the set

$$\{x \mid \frac{2n - 5 + \sqrt{4n^2 - 20n + 49}}{6} \leq x \leq +\infty\}, \tag{8}$$

while, the solutions of the second equality of (8) are elements of the set

$$\{x \mid 0 \leq x \leq \frac{2n + 1 + \sqrt{4n^2 - 20n + 49}}{6}\} \tag{9}$$

From (8) and (9) it is seen directly, that the solutions of the system (7) (and therefore, of the system (6)) are the elements of the set

$$\{x \mid \frac{2n - 5 + \sqrt{4n^2 - 20n + 49}}{6} \leq x \leq \frac{2n + 1 + \sqrt{4n^2 - 20n + 49}}{6}\}.$$

This set contains the unique element

$$k_0 = [\frac{2n + 1 + \sqrt{4n^2 - 20n + 49}}{6}],$$

if $\frac{2n - 5 + \sqrt{4n^2 - 20n + 49}}{6}$ is not a natural number, and the two elements

$$k_1 = [\frac{2n - 5 + \sqrt{4n^2 - 20n + 49}}{6}],$$

$$k_2 = [\frac{2n + 1 + \sqrt{4n^2 - 20n + 49}}{6}],$$

if $\frac{2n - 5 + \sqrt{4n^2 - 20n + 49}}{6}$ is a natural number.

One of the conditions, that the last number to be a natural one is that

$$4n^2 - 20n + 49 = a^2$$

for some natural number a. Therefore,

$$n = \frac{5 \pm \sqrt{a^2 - 24}}{2}.$$

Let $a^2 - 24 = b^2$ for some natural number b. Obviously, $a > b$. Therefore,

$$(a-b)(a+b) = 24$$

and

$$\begin{cases} a - b = 2 \\ a + b = 12 \end{cases}$$

or

$$\begin{cases} a - b = 4 \\ a + b = 6 \end{cases}$$

In the first case we obtain sequentially: $a = 7, b = 5, n = 5, k_1 = 2, k_3 = 3$, but from the condition $n - k \leq 3$ it follows that k_2 cannot be a solution of (6) and hence (6) has only one solution. Analogically, in the second case we obtain: $a = 5, b = 1, n = 2$ or 3 and obviously, (6) has only one solution.

Therefore, the maximal element of the sequence $\{m_k\}_{k=1}^{n-3}$ is $[\frac{2n + 1 + \sqrt{4n^2 - 20n + 49}}{6}]$.