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Associated Legendre Polynomials and Morgan-Vojce Polynomials

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1. INTRODUCTION

Sets $\{X_n\}$

A general class of polynomial sets $\{X_n(x)\}$ is defined recursively by $(X_n(x) \equiv X_n)$

$$X_n = (x+2)X_{n-1} - X_{n-2} \tag{1.1}$$

with

$$X_0 = a, X_1 = b \quad (a, b \text{ integers}). \tag{1.2}$$

Particular cases arise in the following ways:

	X_n	a	b
(B)	B_n	0	1
(b)	b_n	1	1
(C)	C_n	2	$2+x$
(c)	c_n	-1	1

(1.3)

Cases $(B), (b)$ give the Morgan-Voyce polynomials whilst $(C), (c)$ produce polynomials closely related to them. Detailed features of polynomials $(B), (b), (C), (c)$ are developed in [1].

Associated Legendre Polynomials

Riordan [3] derives some properties of, in his nomenclature, the *associated Legendre polynomials* $\rho_n(x)$ and related polynomials $\pi_n(x)$. In our notation, we find that

$$\rho_n(x) = b_{n+1}(x) \tag{1.4}$$

and

$$\pi_n(x) = B_{n+1}(x). \tag{1.5}$$

Chebyshev Polynomials

From [1], we know that if $U_n(x)$ and $T_n(x)$ are Chebyshev polynomials then

$$B_n(x) = U_n\left(\frac{x+2}{2}\right), \tag{1.6}$$

$$b_n(x) = U_n\left(\frac{x+2}{2}\right) - U_{n-1}\left(\frac{x+2}{2}\right), \tag{1.7}$$

$$C_n(x) = 2T_n\left(\frac{x+2}{2}\right), \tag{1.8}$$

$$c_n(x) = U_n\left(\frac{x+2}{2}\right) + U_{n-1}\left(\frac{x+2}{2}\right). \tag{1.9}$$

Chebyshev polynomials $U_n(x)$ and $T_n(x)$ are orthogonal polynomials associated with the interval $(-1,1)$ with weight functions $(1-x^2)^{\frac{1}{2}}$ and $(1-x^2)^{-\frac{1}{2}}$ respectively. Wherever it is sensible to do so, one might hereafter make a mental connection between the Morgan-Voyce polynomials and the Chebyshev polynomials.

Fibonacci and Lucas Numbers

Immediately from [1, (4.1) - (4.4)] with $x = 1$, we obtain

$$B_n(1) = F_{2n}, \tag{1.10}$$

$$b_n(1) = F_{2n+1}, \tag{1.11}$$

$$C_n(1) = L_{2n}, \tag{1.12}$$

$$c_n(1) = L_{2n-1} \tag{1.13}$$

where F_n and L_n are the nth Fibonacci and Lucas numbers, respectively.

Equation (1.11) tell us that, when $x = 1$, Riordan's associated Legendre polynomials transform into odd Fibonacci numbers.

Purpose of this paper

Our objectives here are twofold:

(I) to examine some of the results in [3] from a different perspective, by means of [1], and

(II) to extend Riordan's results, where applicable, to (C) and (c).

2. REQUISITE BACKGROUND

From [1] we reproduce in summary some basic facts.

Generating Functions

$$B_{n+1}(x) = \sum_{k=0}^{n} \binom{n+1+k}{2k+1} x^k, \tag{2.1}$$

$$b_{n+1}(x) = \sum_{k=0}^{n} \binom{n+k}{2k} x^k, \tag{2.2}$$

$$C_{n+1}(x) = \sum_{k=0}^{n} \frac{2n}{n+1-k} \binom{n+k}{n-k} x^k + x^{n+1}, \tag{2.3}$$

$$c_{n+1}(x) = \sum_{k=1}^{n+1} \frac{2n+1}{2k-1} \binom{n+k-1}{n-1+k} x^{k-1}. \tag{2.4}$$

Binet Forms

$$B_n(x) = \frac{\alpha^n - \beta^n}{\Delta}, \tag{2.5}$$

$$b_n(x) = \frac{(1-\beta)\alpha^n - (1-\alpha)\beta^n}{\Delta} = B_n(x) - B_{n-1}(x), \tag{2.6}$$

$$C_n(x) = \alpha^n + \beta^n, \tag{2.7}$$

$$c_n(x) = \frac{(1+\beta)\alpha^n - (1+\alpha)\beta^n}{\Delta} = B_n(x) + B_{n-1}(x), \tag{2.8}$$

where

$$\alpha = \frac{x+2+\sqrt{4x+x^2}}{2}, \ \beta = \frac{x+2-\sqrt{4x+x^2}}{2}, \tag{2.9}$$

so that

$$\alpha\beta = 1, \alpha+\beta = x+2, \alpha-\beta = \sqrt{4x+x^2} = \Delta = \sqrt{(x+2)^2-4}. \tag{2.10}$$

Combinatorial Forms

$$B(x,y) \equiv \sum_{i=1}^{\infty} B_i(x)y^{i-1} = \left[1-\overline{(2+x)y-y^2}\right]^{-1} \equiv B, \tag{2.11}$$

$$b(x,y) \equiv \sum_{i=1}^{\infty} b_i(x)y^{i-1} = (1-y)\left[1-\overline{(2+x)y-y^2}\right]^{-1} \equiv b, \tag{2.12}$$

$$C(x,y) \equiv \sum_{i=1}^{\infty} C_i(x)y^{i-1} = (2+x-2y)\left[1-\overline{(2+x)y-y^2}\right]^{-1} \equiv C, \tag{2.13}$$

$$c(x,y) \equiv \sum_{i=1}^{\infty} c_i(x)y^{i-1} = (1+y)\left[1-\overline{(2+x)y-y^2}\right]^{-1} \equiv c, \tag{2.14}$$

whence

$$b = (1-y)B = 2B - c, \tag{2.15}$$

$$C = (2+x-2y)B = 2(1-y)B + xB = 2b + xB, \tag{2.16}$$

$$c = (1+y)B = [2-(1-y)]B = 2B - b. \tag{2.17}$$

Corresponding to the symbolism [3]

$$f(y) = (1-y-y^2)^{-1}, \tag{2.18}$$

the generating function for Fibonacci numbers, we introduce the notation

$$g(y) = (1-y+y^2)^{-1}. \tag{2.19}$$

Incidentally in passing, the function

$$f \equiv f(x,y) = 1-(2+x)y+y^2 \tag{2.20}$$

satisfies the partial differential equation

$$\left(1+\frac{x}{2}-y\right)\frac{\partial f}{\partial x}-y\ \frac{\partial f}{\partial y}=0. \tag{2.21}$$

3. SOME NEW PROPERTIES OF $\{X_n(x)\}$.

From (2.15)–(2.17) we deduce, e.g., that symbolically

$$\begin{aligned} c+b &= 2B, & (3.1)\\ C-2b &= xB, & (3.2)\\ c-b &= 2yB, & (3.3)\\ bc &= (1-y^2)B^2 = B^2-(yB)^2, & (3.4)\\ c^2+b^2 &= 2(1+y^2)B^2 = 2\left[B^2+(yB)^2\right], & (3.5)\\ c^2-b^2 &= 4B\cdot yB. & (3.6)\end{aligned}$$

Instances of these identities are, e.g., cf. [1],

$$\begin{aligned} C_3(x)-2b_3(x) &= x(3+4x+x^2) = xB_3(x),\\ b_3(x)c_3(x) &= 5+20x+21x^2+8x^3+x^4 = B_3^2(x)-B_2^2(x) \quad \text{(note)},\\ c_2^2(x)+b_2^2(x) &= 2[1+4+4x+x^2] = 2[B_2^2(x)+B_1^2(x)],\\ c_2^2(x)-b_2^2(x) &= 4(2+x) = 4B_2(x)\cdot B_1(x).\end{aligned}$$

Observe the mild subtlety occurring in (3.3)–(3.6), namely, that the existence of the factor y with B necessitates a reduction by 1 of the subscript in the corresponding polynomial.

4. $\mathbf{B_n(x)}$ AND $\mathbf{b_n(x)}$

Firstly, we quote two results form [3,pp.88-89], in our notation.

Theorem 1: $B_{n+1}(x)=\sum_{k=0}^{[\frac{n}{2}]}\binom{n-k}{k}(2+x)^{n-2k}(-1)^k.$

Proof: Proceed from (1.1) by induction on n and apply Pascal's (combinatorial) formula with a little algebraic refinement.

Theorem 2: $B_{n+2}(x)=\sum_{k=0}^{[\frac{n}{2}]}(2+x)^{n-2k}2^{-n+2k}(-1)^k\sum_{j=k}^{[\frac{n}{2}]}\binom{n+1}{2j+1}\binom{j}{k}.$

Proof: Start from (2.5) and re-arrange terms.

Corollary 1: $\sum_{j=k}^{[\frac{n}{2}]} \binom{n+1}{2j+1}\binom{j}{k} = 2^{n-2k}\binom{n-k}{k}, \qquad k = 0, 1, \ldots, \left[\frac{n}{2}\right].$

Proof: Combine Theorems 1 and 2, taking $x = -1$.

Rivlin [4, p.35] lists Theorem 1 in a simplified form as Exercise 1.5.13, where (1.6) underpins the relationship. Also see [2,p.257].

Now, from (2.11), following [3] we obtain

$$
\begin{aligned}
B(x,0) &= 1, & (4.1)\\
B(0,y) &= (1-y)^{-2} = \sum_{n=1}^{\infty} n y^{n-1} \left[= (b(0,y))^2, (4.11)\right], & (4.2)\\
B(1,y) &= (1-3y+y^2)^{-1}, & (4.3)\\
B(1,y^2) &= (1-3y^2+y^4)^{-1} = \sum_{n=0}^{\infty} F_{2n+2} y^{2n} \text{ (cf.(1.10))} & (4.4)\\
&= y^{-1}\cdot\frac{1}{2}[f(y) - f(-y)] \text{ by (2.18).} & (4.4a)
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
B(-1,y) &= (1-y+y^2)^{-1} = (1+y)(1+y^3)^{-1}\\
&= \sum_{n=0}^{\infty}(-1)^n(y^{3n} + y^{3n+1} + 0\cdot y^{3n+2}), & (4.5)
\end{aligned}
$$

whence, with (2.1), Theorem 1, $x = -1$, and $n \longrightarrow 3n$, $n \longrightarrow 3n+1$, $n \longrightarrow 3n+2$ in turn, we deduce by comparing coefficients [3] that

$$\sum_{k=0}^{3n}\binom{3n+k+1}{2k+1}(-1)^k = \sum_{k=0}^{[\frac{3n}{2}]}\binom{3n-k}{k}(-1)^k = (-1)^n = B_{3n+1}(-1), \qquad (4.6)$$

$$\sum_{k=0}^{3n+1}\binom{3n+k+2}{2k+1}(-1)^k = \sum_{k=0}^{[\frac{3n+1}{2}]}\binom{3n+1-k}{k}(-1)^k = (-1)^n = B_{3n+2}(-1), \qquad (4.7)$$

$$\sum_{k=0}^{3n+2}\binom{3n+k+3}{2k+1}(-1)^k = \sum_{k=0}^{[\frac{3n+2}{2}]}\binom{3n+2-k}{k}(-1)^k = 0 = B_{3n+3}(-1). \qquad (4.8)$$

Hence, $x+1$ is a zero of $B_{3n+3}(x)$.

Combining (2.1), (1.10) and Theorem 1 yields

$$\sum_{k=0}^{\infty}\binom{n+k+1}{2k+1} = \sum_{k=0}^{\infty}(-1)^k\binom{n-k}{k}3^{n-2k} = B_{n+1}(1) = F_{2n+2}. \qquad (4.9)$$

Coming now to $b_n(x)$ in (2.12), we derive [3]

$$
\begin{aligned}
b(x,0) &= 1, & (4.10)\\
b(0,y) &= (1-y)^{-1} = \sum_{n=0}^{\infty} y^n, & (4.11)
\end{aligned}
$$

$$b(1,y) = (1-y)(1-3y+y^2)^{-1} = \frac{1}{2}[f(\sqrt{y}) + f(-\sqrt{y})], \tag{4.12}$$

$$b(1,y^2) = (1-y^2)(1-3y^2+y^4)^{-1} = \sum_{n=0}^{\infty} F_{2n+1}y^{2n} \text{ (cf. (1.11))}. \tag{4.13}$$

Moreover, (2.12) leads to

$$b(-1,y) = (1-y)(1-y+y^2)^{-1} = (1-y^2)(1+y^3)^{-1} \tag{4.14}$$

$$= \sum_{n=0}^{\infty}\left[(-1)^n y^{3n} + 0\cdot y^{3n+1} + (-1)^{n+1}y^{3n+2}\right] \tag{4.14a}$$

whence, by (2.2)

$$b_{3n+1}(-1) = \sum_{k=0}^{3n}\binom{3n+k}{2k}(-1)^n = (-1)^n, \tag{4.15}$$

$$b_{3n+2}(-1) = \sum_{k=0}^{3n+1}\binom{3n+1+k}{2k}(-1)^n = 0, \tag{4.16}$$

$$b_{3n+3}(-1) = \sum_{k=0}^{3n+2}\binom{3n+2+k}{2k}(-1)^n = (-1)^{n+1}, \tag{4.17}$$

in accordance with [3]. For example, (4.17) yields $b_6(-1) = \sum_{k=0}^{5}\binom{5+k}{2k}(-1)^k = 1$ in concurrence with $b_6(x) = 1 + 15x + 35x^2 + 28x^3 + 9x^4 + x^5$ at $x = -1$. Notice that $x+1$ is a zero of $b_{3n+1}(x)$.

Invoking (2.19), we readily calculate from (4.12) that

$$b(-1,y) = \frac{1}{2}[g(i\sqrt{y}) + g(-i\sqrt{y})] \qquad (i^2 = -1). \tag{4.18}$$

5. $\mathbf{C_n(x)}$ AND $\mathbf{c_n(x)}$.

Turning next to (2.13), we have immediately that

$$C(x,0) = 2 + x \tag{5.1}$$

$$C(0,0) = 2 \tag{5.2}$$

$$C(0,y) = 2(1-y)^{-1} = 2\sum_{n=0}^{\infty} y^n = 2b(0,y) \tag{5.3}$$

$$C(1,y) = (3-2y)\left[1-3y+y^2\right]^{-1} \tag{5.4}$$

while

$$C(-1,y) = (1-2y)(1+y)(1+y^3)^{-1} = (1-y-2y^2)(1+y^3)^{-1} \tag{5.5}$$

$$= \sum_{n=0}^{\infty}\left[(-1)^n y^{3n} + (-1)^{n+1}y^{3n+1} + 2(-1)^{n+1}y^{3n+2}\right] \tag{5.5a}$$

whence

$$\begin{aligned}
C_{3n+1}(-1) &= (-1)^n = 2\sum_{k=0}^{3n} \frac{3n+1}{3n+1-k}\binom{3n+k}{3n-k}(-1)^k + (-1)^{n+1}, && (5.6)\\
C_{3n+2}(-1) &= (-1)^{n+1} = 2\sum_{k=0}^{3n+1} \frac{3n+2}{3n+2-k}\binom{3n+1+k}{3n+1-k}(-1)^k + (-1)^n, && (5.7)\\
C_{3n+3}(-1) &= 2(-1)^{n+1} = 2\sum_{k=0}^{3n+2} \frac{3n+3}{3n+3-k}\binom{3n+2+k}{3n+2-k}(-1)^k + (-1)^{n+1}. && (5.8)
\end{aligned}$$

So, $C_4(-1) = -1$ in accord with $C_4(x) = 2 + 16x + 20x^2 + 8x^3 + x^4$ when $x = -1$. Equations in the second and third columns of (5.6)–(5.8) can be slightly simplified by transferring the term in $(-1)^n$. Obviously, $C_{3n+2}(-1) = \frac{1}{2}C_{3n+3}(-1) = -C_{3n+1}(-1)$.

One may also confirm that

$$C(1, y^2) = (3 - 2y^2)[1 - 3y^2 + y^4]^{-1} = \sum_{n=0}^{\infty} L_{2n+2}y^{2n} \quad \text{(cf. (1.12))}. \tag{5.9}$$

Lastly, (2.14) reveals that

$$\begin{aligned}
c(x, 0) &= 1, && (5.10)\\
c(0, y) &= (1 + y)(1 - y^2)^{-1} = (1 - y)^{-1} = b(0, y), && (5.11)\\
c(1, y) &= (1 + y)[1 - 3y + y^2]^{-1}, && (5.12)
\end{aligned}$$

whereas

$$\begin{aligned}
c(-1, y) &= (1 + y)[1 - y + y^2]^{-1} = (1 + y)^2(1 + y^3)^{-1} && (5.13)\\
&= \sum_{n=0}^{\infty}[(-1)^n y^{3n} + 2(-1)^n y^{3n+1} + (-1)^n y^{3n+2}] && (5.13a)
\end{aligned}$$

generating

$$\begin{aligned}
c_{3n+1}(-1) &= (-1)^n = \sum_{k=1}^{3n+1} \frac{6n+1}{2k-1}\binom{3n+k-1}{3n-k+1}(-1)^{k-1}, && (5.14)\\
c_{3n+2}(-1) &= 2(-1)^n = \sum_{k=1}^{3n+2} \frac{6n+3}{2k-1}\binom{3n+k}{3n-k+2}(-1)^{k-1}, && (5.15)\\
c_{3n+3}(-1) &= (-1)^n = \sum_{k=1}^{3n+3} \frac{6n+5}{2k-1}\binom{3n+k+1}{3n-k+3}(-1)^{k-1}. && (5.16)
\end{aligned}$$

That is, for example, $c_5(-1) = -2 = [9 + 30x + 27x^2 + 9x^3 + x^4]_{x=-1}$. In summary, $c_{3n+1}(-1) = \frac{1}{2}c_{3n+2}(-1) = c_{n+3}(-1)$.

Ultimately, it follows that

$$c(1, y^2) = (1 + y^2)[1 - 3y^2 - y^4]^{-1} = \sum_{n=0}^{\infty} L_{2n+1}y^{2n} \quad \text{(cf.(1.13))}. \tag{5.17}$$

Summary for $X_m(-1)$

Collecting the data for $X_m(-1)$ - refer to (1.3) - we have the ensuing striking tabular information, writing $(-1)^n = k$,

$X_m(-1)$ \ m	$3n$	$3n+1$	$3n+2$
$B_m(-1)$	0	k	k
$b_m(-1)$	k	k	0
$C_m(-1$	$2k$	k	$-k$
$c_m(-1)$	$-k$	k	$2k.$

(5.18)

Special features of this table are

(i) $X_{3n+1}(-1) = (-1)^n$,

(ii) the interchanges $B \leftrightarrow b, C \leftrightarrow c$ for column 1 $\leftrightarrow$ column 3,

(iii) $\sum_{i=0}^{2} X_{3n+i}(-1) = 2k$,

(iv) $\sum_{i=0}^{2}(-1)^i X_{3n+i}(-1) = 0$,
i.e., $X_{3n+2}(-1) = X_{3n+1}(-1) - X_{3n}(-1)$ (recurrence).

Periodicity occurring for $\{X_m(-1)\}$ is mentioned in [1], along with other numerical values arising form the set $\{X_m(x)\}$. More detailed information on this numerical aspect is to be found in [5].

From (iv), $X_{3n+1}(-1) = X_{3n}(-1) + X_{3n+2}(-1)$ which is a typically important result in our investigation of three successive terms of more general polynomial sequences. See (6.4).

Setting $x = -1$ in (1.3), we always have $X_1(= b) = 1$ leading to the specific values ± 1 for $X_{3n+1}(-1)$ in (i).

6. AFTERMATH

It would be richly rewarding if compact results for $b_n(x), C_n(x)$, and $c_n(x)$ involving powers of $2 + x$ corresponding to these for $B_n(x)$ (especially that in Theorem 1) could be discovered, but such properties, if they exist, are currently elusive. However, a partial achievement is probably better than no achievement at all.

Now

$$\begin{aligned} b_{n+1}(x) &= (1+x)B_n(x) - B_{n-1}(x) \qquad \text{by } [1,(3.29)] \\ &= (2+x)B_n(x) - c_n(x) \qquad [1,(3.7)] \\ &= \sum_{k=0}^{[\frac{n-1}{2}]} \binom{n-1-k}{k} (2+x)^{n-2k}(-1)^k - \sum_{k=1}^{n} \frac{2n-1}{2k-1} \binom{n+k-2}{n-k} x^{k-1} \quad (6.1) \end{aligned}$$

by [1, (3.23)]. The first portion of (6.1) contains powers of $2+x$; the second portion does not, though $c_n(x)$ is the sum of two such expressions.

Next

$$\begin{aligned} C_n(x) &= (2+x)B_n(x) - 2B_{n-1} \qquad \text{by } [1, (3.31)] \\ &= \sum_{k=0}^{[\frac{n-1}{2}]} \binom{n-1-k}{k} (2+x)^{n-2k}(-1)^k - 2 \sum_{k=0}^{[\frac{n-2}{2}]} \\ &\quad \binom{n-2-k}{k} (2+x)^{n-2-2k}(-1)^k \qquad (6.2) \end{aligned}$$

while

$$\begin{aligned} c_{n+1}(x) &= (3+x)B_n(x) - B_{n-1}(x) \qquad \text{by } [1, (3.30)] \\ &= (2+x)B_n(x) + b_n(x) \qquad \text{by } [1, (2.13)] \\ &= \sum_{k=0}^{[\frac{n-1}{2}]} \binom{n-1-k}{k} (2+x)^{n-2k}(-1)^k + b_n(x) \qquad (6.3) \end{aligned}$$

where $b_n(x)$ is derived from (6.1) by adjustment.

Thus, each of (6.1), (6.2), and (6.3) is expressible as sums of functions involving $2+x$ which seemingly do not simplify into a single condensed form. This may be as good as it gets. One should be thankful for small mercies.

Perhaps it is significant that Riordan [3] does not offer any compact Theorem 1 counterpart of $B_{n+1}(x)$ for his associated Legendre polynomials $b_{n+1}(x)$.

Despite the limited accomplishment in (6.1)-(6.3), some comfort can be gleaned by considering a useful polynomial defined in terms of $B_{n+1}(x)$ and $B_{n-1}(x)$.

Suppose, then in conclusion, we introduce the polynomial

$$B_n^*(x) = B_{n+1}(x) + B_{n-1}(x). \qquad (6.4)$$

Accordingly,

$$B_n^*(x) = \sum_{k=0}^{[\frac{n}{2}]} \binom{n-k}{k} (2+x)^{n-2k}(-1)^k$$

$$+ \sum_{k=0}^{[\frac{n-2}{2}]} \binom{n-2-k}{k} (2+x)^{n-2-2k}(-1)^k \qquad \text{by Theorem 1}$$

$$= \sum_{k=0}^{[\frac{n}{2}]} \binom{n-1-2k}{k} (2+x)^{n-2k}(-1)^k \tag{6.5}$$

on expansion and employment of Pascal's formula, coupled with a little algebraic manipulation. Our expression (6.5) is now precisely analogous to that in Theorem 1. Similar, though less elegant, results flow from $b_n^*(x)$, $C_n^*(x)$, and $c_n^*(x)$ defined as in (6.4), with appropriate algebraic maneuvering.

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