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Strong Bertrand's postulate revisited

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Let us consider a positive integer k , and denote by d_k the least positive integer n which has the property $p_{n-1} < 2p_n - k$ where p_n is the n-th prime number.

As $p_n > 2p_n - p_{n+1} > k$, it follows that $n > \pi(k)$, so $d_k > \pi(k)$ is the number of primes not exceding k .

In [1] , it is proved that $\pi(k) > \frac{k}{\log k}$ for $k \geq 17$, hence $d_k > \frac{k}{\ln k}$ for $k \geq 17$.

The converse problem i.e. finding upperbounds for d_k using elementary tools only, was studied by Udrescu [4] who proved that $d_k < \exp(1 + \exp(k+10))^{\frac{1}{2}}$ and by Sándor [3] who proved that $d_k \sim \frac{k}{\ln k}$ and $d_k \leq \left[\frac{13}{12} \cdot \frac{k}{\log k - \log\log k}\right] + 1$ for $k \geq 4$.

Using "strong" results i.e. based on non-elementary methods, we shall obtain, in this note, a better upper bound than the above mentioned ones.

We shall use the Rosser – Shoenfeld inequalities [2] : for $n \geq 20$,

$$p_n < n\left(\log n + \log\log n - \frac{3}{2}\right) \tag{1}$$

and Robin's inequality [1], for $n \geq 2$,

$$p_n > n(\log n + \log\log n - a) \tag{2}$$

where a=1,0077629.

Our main result is

Theorem. For $k \geq 10$ we have $d_k \leq \frac{k}{\log k - 2}$.

In order to proof this theorem we shall use the following

Lemma. For $n \geq 10$,

$$p_{n-1} - p_n < 0.6n \tag{3}$$

Prof. Using (1) and (2), we have, for $n \geq 19$,

$$p_{n+1} - p_n < 0.50077629n + \log(n+1) + \log\log(n+1) + n\log\left(1+\frac{1}{n}\right) + n\log\frac{\log(n+1)}{\log n} - 0.5 <$$

$$< 0.50077629n + \log(n+1) + \log\log(n+1) + 0.5 + \log\frac{1}{n}$$

as $\log(1+x) \le x$ for $x > -1$.

In order to prove that $p_{n+1} - p_n < 0.6n$ it is suffices to prove that:

$$0.09922371n - \log(n+1) - \log\log(n+1) - \frac{1}{\log n} - 0.5 > 0.$$

Let consider $f(x) = 0.099x - \log(x+1) - \log\lg(x+1) - \frac{1}{\log n} - 0.5$, for $x \ge 19$;

we obtain: $f'(x) = 0.099 - \frac{1}{x+1} - \frac{1}{(x+1)\ln(x+1)} + \frac{1}{x\log^2 x} > 0.099 - \frac{1}{20} - \frac{1}{20\log 20}$,

because $x+1 \ge 20$, hence for $x \ge 19$, $f'(x) > 0$, i.e. f is increasing.

We have $f(65) > 6.435 - 5.622 - 0.239 - 0.5 > 0$ that is $p_{n+1} - p_n < 0.6n$, for $n \ge 65$. A simple computation shows that inequality (3) is true for $n \ge 10$.

Proof of the theorem. Using (3) and (2) it follows:

$$2p_n - p_{n+1} > p_n - 0.6 > n(\log n + \log\log n - a - 0.6) \qquad (4)$$

for $n \ge 10$.

Let be $g(x) = x - 10\log x + 20$ for $x \ge 10$. We obtain $g'(x) = \frac{x-10}{x} \ge 0$ hence g is increasing. As $g(10) = 10(3 - 2.31) > 0$, hence $g(x) > 0$ for $x \ge 10$.

It follows that, for $k \ge 10$, we have $\frac{k}{\log k - 2} > 10$ and, for

$$n \ge \frac{k}{\log k - 2}, 2p_n - p_{n-1} \ge \frac{k}{\log k - 2}\left(\log\frac{k}{\log k - 2} + \log\log\frac{k}{\log k - 2} - a - 0.6\right) =$$

$$= \frac{k}{\log k - 2}\left(\log k - 2 + 1.4 - a - \log\frac{\log k - 2}{\log k - \log(\log k - 2)}\right).$$

We shall obtain $2p_n - p_{n-1} > k$ providing that:

$$1.4 - a > \log\frac{\log k - 2}{\log k - \log(\log k - 2)} \qquad (5)$$

Denote $\log k = x \ge \log 10$ and we have to prove that $e^{1.4-a}(x - \log(x-2)) > x - 2$.

As $e^{1.4-a} > 1.48$, we will consider $h(x) = 0.48x - 1.48\log(x-2) + 2$, and as $h'(x) = \frac{0.48x - 2.44}{x-2}$ the lowest value of $h(x)$ is reached for $x_0 = \frac{61}{12}$. We have $h(x_0) = 2.44 - 1.66 + 2 > 0$ hence $h(x) > 0$ i.e. (5) is true for $k \geq 10$.

We proved that, for $k \geq 10$ and $n \geq \frac{k}{\log k - 2}$, we have $2p_n - p_{n+1} > k$ that is

$d_k \leq \frac{k}{\log k - 2}$.

So, the Sándor's statement: $d_k \sim \frac{k}{\log k}$ takes the following precise form, for

$k \geq 1$:

$$\frac{k}{\log k - 2} > d_k > \frac{k}{\log k}.$$

References

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