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A SIMPLE PROOF THAT THE AREA OF A PYTHAGOREAN TRIANGLE IS SQUARE-FREE

J.V. Leyendekkers
The University of Sydney, 2006, Australia

A.G. Shannon
University of Technology, Sydney, 2007, &
KvB Institute of Technology, North Sydney, 2060, Australia

Abstract

By using the geometry of a pythagorean triangle with a circle inscribed, it can be proved by a simple geometric proof that the area of such a triangle can never be a square. The class structure of the modular ring $\mathbb{Z}_4$ can be used to illustrate the result for various Pythagorean triples.

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1. Introduction

By using his method of infinite descent, Fermat was able to prove that the area of a Pythagorean triangle can never be a square [1]. This result has also been illustrated in the context of the structure of the modular rings $\mathbb{Z}_4$ and $\mathbb{Z}_6$ [3]. In the present paper we provide a simple trigonometric proof by utilising the properties of an incircle of right triangle.

2. The Square Free Nature of the Area of the Right Triangle

Let the area of the triangle ABC be represented by S and the radius of the inscribed circle by r with centre at O. Then

$$S = \Delta BOC + \Delta COA + \Delta AOB$$

$$= \tfrac{1}{2}r(a+b+c). \tag{2.1}$$

The straight lines OA, OB, OC bisect the angles A, B, C respectively, so that

$$a = r(cot(B/2) + cot(C/2)) \tag{2.2}$$

$$b = r(cot(C/2) + cot(A/2)) \tag{2.3}$$

$$c = r(cot(A/2) + cot(B/2)) \tag{2.4}$$

Hence [4:78],

$$a+b+c = 2r(cot(A/2) + cot(B/2) + cot(C/2))$$

and

$$S = r^2(cot(A/2) + cot(B/2) + cot(C/2)). \tag{2.5}$$

But

$$\tfrac{1}{2}(A+B+C) = 90^o,$$

so

$$cot(A/2) + cot(B/2) + cot(C/2) = cot(A/2)cot(B/2)cot(C/2), \tag{2.6}$$

and

$$S = r^2(cot(A/2)cot(B/2)) \tag{2.7}$$

since $\angle C = 90^0$.

Then, using Equation (2.6), we get

$$S = r^2(x(x+1)/(x-1)) \tag{2.8}$$

with $x \in \{cot(A/2), cot(B/2)\}$.

Because of its form

$$f(x) = \frac{x(x+1)}{x-1}$$

will not be an integer except for $1 < x \leq 3$, when $f(x) = 6$. If the denominator is a square, say m^2, then $x = 1 + m^2$ and the numerator becomes $(m^2+1)(m^2+2)$, which cannot be a square. Hence, S cannot be a square.

3. Within the Modular Ring $\mathbb{Z}_4$

$\mathbb{Z}_4$ has four equivalence classes, i, and the integers N are given by

$$N = 4R_i + i \tag{3.1}$$

where R_i is the row containing N, $i \in \{0, 1, 2, 3\}$ as in Table 3.1.

Class	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$
Row, R	$4R_0$	$4R_1+1$	$4R_2+2$	$4R_3+3$
0	0	1	2	3
1	4	5	6	7
2	8	9	10	11
3	12	13	14	15

Table 3.1

Since $cot(A/2) = s/(s-b)$, $cot(B/2) = (s-b)/(s-c)$, with $s = \frac{1}{2}(a+b+c)$ [4],

$$S = r^2(a+b+c)/(a+b-c). \tag{3.2}$$

But $(a+b-c) = 2r$ and $(a+b+c) = 2(r+c)$ so that

$$S = r^2 + cr. \tag{3.3}$$

As shown previously [3], S must be even and in class $\overline{0}$, if it is a square. Hence, $S = 4R_0$ in this case. On the other hand, c is odd and in class $\overline{1}$. If S is a square in $\overline{0}$, then R_0 must be a square. Thus the two class sets we need to consider for Pythagorean triples $< c, b, a >$ are $< \overline{1}, \overline{0}, \overline{1} >$ and $< \overline{1}, \overline{0}, \overline{3} >$ [3].

3.1 r odd.

This will occur in $< \overline{1}, \overline{0}, \overline{3} >$. If $r \in \overline{1}$, then $S \in \overline{2}$ which contains no squares. Hence, $r \in \overline{3}$ if S is a square. Substitution of $r = 4R_3 + 3$ and $c = 4R_1 + 1$ into Equation (3.3) yields

$$R_0 = 4(R_3^2 + R_1R_3 + \tfrac{1}{4}(7R_3 + 3R_1)) + 3 \quad (3.4)$$

$$R_0 = 4(R_3^2 + R_1R_3 + \tfrac{1}{4}(7R_3 + 3R_1 + 1)) + 2 \quad (3.5)$$

$$R_0 = 4(R_3^2 + R_1R_3 + \tfrac{1}{4}(7R_3 + 3R_1 + 2)) + 1 \quad (3.6)$$

$$R_0 = 4(R_3^2 + R_1R_3 + \tfrac{1}{4}(7R_3 + 3R_1 + 3)) \quad (3.7)$$

Equations (3.4) and (3.5) cannot give squares as Classes $\overline{2}$ and $\overline{3}$ do not contain squares, so Equations (3.6) and (3.7) are the only valid ones if the area in question is a square. In this case, when R_0 is odd it lies in the row $R_3^2 + R_1R_3 + \frac{1}{4}(7R_3 + 3R_1 + 2)$ which must be even if R_0 is a square, and R_1 and R_3 must have the same parity for integer solutions. When R_0 is even and a square it will be given by Equation (3.7), and the row $R_3^2 + R_1R_3 + \frac{1}{4}(7R_3 + 3R_1 + 3)$ has to be a square with R_1 and R_3 of opposite parity.

Table 3.1 shows the characteristics of a few triples in , $< \overline{1}, \overline{0}, \overline{3} >$. For instance, with $r \in \overline{1}$, $S \in \overline{2}$ which contains no squares. For triples with $r \in \overline{3}$ and $S \in \overline{0}$, most have $R_0 \in \{\overline{2}, \overline{3}\}$ which contains no squares.

Triple	r	$r \in$	S	$S \in$	R_0	$R_0 \in$	$f(x)$
$< 5, 4, 3 >$	1	$\overline{1}$	6	$\overline{2}$			6
$< 37, 35, 12 >$	5	$\overline{1}$	210	$\overline{2}$			42/5
$< 17, 15, 8 >$	3	$\overline{3}$	60	$\overline{0}$	15	$\overline{3}$	20/3
$< 65, 63, 16 >$	7	$\overline{3}$	504	$\overline{0}$	126	$\overline{2}$	72/7
$< 25, 24, 7 >$	3	$\overline{3}$	84	$\overline{0}$	21	$\overline{1}$	28/3
$< 305, 224, 207 >$	63	$\overline{3}$	23184	$\overline{0}$	5796	$\overline{0}$	368/63

Table 3.2: $< \overline{1}, \overline{0}, \overline{3} >$

3.2 r even.

This will occur in $< \overline{1}, \overline{0}, \overline{1} >$. If $r \in \overline{2}$, $S \in \overline{2}$ as well and cannot be a square. With $r \in \overline{0}$, and in a row of Table 3.1 that falls in $\{\overline{2}, \overline{3}\}$, again R_0 cannot be a square. Some of these triples are illustrated in Table 3.3.

Triple	r	$r \in$	S	$S \in$	R_0	$R_0 \in$	$f(x)$
$< 13, 12, 5 >$	2	$\overline{2}$	30	$\overline{2}$			$\frac{15}{2}$
$< 85, 77, 36 >$	14	$\overline{2}$	1386	$\overline{2}$			$\frac{99}{14}$
$< 29, 21, 20 >$	12	$\overline{0}$	492	$\overline{0}$	123	$\overline{3}$	$\frac{41}{12}$
$< 21089, 20961, 2320 >$	1096	$\overline{0}$	24314760	$\overline{0}$	6078690	$\overline{2}$	$\frac{4437}{548}$
$< 32777, 32745, 1448 >$	708	$\overline{0}$	23707380	$\overline{0}$	5926845	$\overline{1}$	$\frac{33485}{708}$
$< 233, 208, 105 >$	80	$\overline{0}$	25040	$\overline{0}$	6260	$\overline{0}$	$\frac{313}{80}$

Table 3.3: $< \overline{1}, \overline{0}, \overline{1} >$

We note by way of conclusion that values for $f(x)$ are given in Tables 3.2 and 3.3 so that we can see if $x \in \mathbb{Z}$ and $(x-1) = m^2$, then

$$f(x) = (1 + m^2)(2 + m^2)/m^2.$$

If m is odd, then $(1 + m^2) \in \overline{2}$ and $(2 + m^2) \in \overline{3}$, so that the numerator falls in $\overline{2}$ and can never be a square. If m is even, then $(1 + m^2) \in \overline{1}$ and $(2 + m^2) \in \overline{2}$, so that again the numerator falls in $\overline{2}$ and cannot be a square. When $x = p/q, p, q \in \mathbb{Z}$, then, assuming the denominator $q(p-q)$ is a square m^2,

$$f(x) = (q^2 + m^2)(2q^2 + m^2)/(mq)^2.$$

For the set $< \overline{1}, \overline{0}, \overline{3} >$ the numerator is even and denominator odd, so m and q are both odd. Hence the numerator falls in Class $\overline{2}$ and cannot be a square; that is,

$$(q^2 + m^2)(2q^2 + m^2) \in (\overline{1} + \overline{1})(\overline{2} \times \overline{1} + \overline{1}) = \overline{2} \times \overline{3} = \overline{2}.$$

For the set $< \overline{1}, \overline{0}, \overline{1} >$ the numerator is odd and the denominator is even, so that q and m must both be even and (m^2/q) odd. In the form given above, the numerator of $f(x)$ now falls in Class $\overline{0}$ but the row cannot be a square; hence the numerator cannot be a square.

Exercises arising from this could involve finding the non-Euclidean analogues [*cf.* 2: Chs.8,9].

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