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ON THE 62-th SMARANDACHE'S PROBLEM

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In [1] Florian Smarandache formulated 105 unsolved problems.

The 62-th problem is the following:

Let $1 \leq a_1 < a_2 < \dots$ be an infinite sequence of integers such that any three members do not constitute an arithmetic progression. Is it true that always

$$\sum_{n \geq 1} \frac{1}{a_n} \leq 2?$$

Here we shall give a counterexample.

Easily it can be seen that the set of numbers $\{1, 2, 4, 5, 10\}$ does not contain three numbers which are members of an arithmetic progression. On the other hand

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{10} = 2\frac{1}{20} > 2.$$

Therefore, Smarandache's problem is not true in the present form, because the sum of the members of every one sequence with the abive property and with first members 1, 2, 4, 5, 10 will be bigger than 2.

Let us consider the set A of the possible minimal natural numbers, that satisfy the condition of the above problem, every one of which is bigger than 1. Its members are obtained by the following formulas

$$a_1 = 2, \ a_2 = 3, \ a_3 = 5, \ a_4 = 6$$

$$a_{4k+1} = 2a_{4k} - 1$$

$$a_{4k+2} = a_{4k+1} + 1$$

$$a_{4k+3} = a_{4k+2} + 2$$

$$a_{4k+4} = a_{4k+3} + 1.$$

Therefore $a_{4k+5} = 2a_{4k+1} + 7$. Hence $\frac{1}{a_{4k+5}} < \frac{1}{2a_{4k+1}}$ and

$$\sum_{a \in A} \frac{1}{a} < \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + 4 \sum_{k=1}^{\infty} \frac{1}{a_{4k+1}} < \frac{36}{30} + \frac{4}{11} \sum_{k=1}^{\infty} \frac{1}{2^k} < \frac{86}{55} = 1.563\ldots < 2.$$

Now we must prove that so constructed sequence $\{a_i\}_{i=1}^{\infty}$ does not contin a three-element subsequence which is an arithmetic progression.

Let us assume that there are three members b, c and d of $\{a_i\}_{i=1}^{\infty}$ and let they be members of an arithmetic progression. Let $b < c < d$. For these members the following four cases are possible:

1. If $b = a_{4k+p}, c = a_{4k+q}, d = a_{4k+r}$, where k, p, q, r are natural numbers and $p, q, r \in \{1, 2, 3, 4\}$, then $p < q < r$. But onbiously, there are not three members of the set $\{2a_{4k} - 1, 2a_{4k}, 2a_{4k} + 2, 2a_{4k} + 3\}$m which are members of an arithmetic progression.

2. If $b = a_{4k+p}, c = a_{4k+q}, d = a_{4l+r}$, where k, l, p, q, r are natural numbers and $k < l$, then

$$\begin{aligned} c - b &\leq (2a_{4k} + 3) - (2a_{4k} - 1) \\ &= 4 < 2a_{4k} + 2 \\ &= 2(2a_{4k} - 1) + 7 - (2a_{4k} + 3) \\ &= 2a_{4k+1} + 7 - (2a_{4k} + 3) \\ &\leq d - c, \end{aligned}$$

i.e., b, c, d cannot be members of an arithmetic progression.

3. If $b = a_{4k+p}, c = a_{4l+q}, d = a_{4l+r}$, where k, l, p, q, r are natural numbers and $k < l$, then (from $a_4 = 6$)

$$\begin{aligned} d - c &\leq (2a_{4l} + 3) - (2a_{4l} - 1) \\ &= 4 < 5 = a_{4k} - 1 \\ &= (2a_{4k} - 1) - a_{4k} \\ &= a_{4k+1} - a_{4k} \\ &= 2a_{4k+1} + 7 - (2a_{4k} + 3) \\ &\leq d - c, \end{aligned}$$

i.e., b, c, d cannot be members of an arithmetic progression.

4. If $b = a_{4k+p}, c = a_{4l+q}, d = a_{4m+r}$, where k, l, m, p, q, r are natural numbers, then (from $a_1 = 2$)

$$\begin{aligned} c - b &\leq (a_{4l+1} + 4) - 2 \\ &= a_{4l+1} + 2 < a_{4l+1} + 3 \\ &= 2a_{4l+1} + 7 - (a_{4l} + 4) \\ &= a_{4l+5} - a_{4l+4} \leq d - c, \end{aligned}$$

i.e., b, c, d cannot be members of an arithmetic progression.

Every other set with the above property has members which are bigger than the corresponding members of the set A. Therefore, the sum of their reciprocal values will be smaller than the sum of the members of A. Hence, the Smarandache's problem will be valid if all elements are bigger that 1.

REFERENCE:

[1] F. Smarandache, Only problems, not solutions!. Xiquan Publ. House, Chicago, 1993.