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ON THE 43-rd AND 44-th SMARANDACHE'S PROBLEMS

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The 43-rd and 44-th problems from [1] are the following (see also Problem 42 from [2]):
(Inferior) factorial part:

$$1,2,2,2,2,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,24,24,24,24,24,24,$$

$$24,24,24,24,24,24,24,24,24,24,24,24,24,24,24,24,24,24,24,24,24,$$

$$24,24,24,24,24,24,24,24,\ldots$$

($F_p(n)$ is the largest factorial less than or equal to n.)

(Superior) factorial part:

$$1,2,6,6,6,6,24,24,24,24,24,24,24,24,24,24,24,24,24,24,24,24,24,24,$$

$$120,120,120,120,120,120,120,120,\ldots$$

($f_p(n)$ is the smallest factorial greater than or equal to n.)
Study these sequences.

Below we shall use the usual notations: $[x]$ and $\lceil x \rceil$ for the integer part of the real number x and for the least integer $\geq x$, respectively.

First, we shall extend the definition of the function "factorial" (it is possible, it is already defined, but the author does not know this). It is defined only for natural numbers and for a given such number n it has the form:

$$n! = 1.2. \ldots .n.$$

Let the new form of the function "factorial" be the following for the real positive number y:

$$y! = y.(y-1).(y-2)\ldots(y-[y]+1),$$

where $[y]$ denotes the integer part of y.

Therefore, for the real number $y > 0$:

$$(y+1)! = y!.(y+1).$$

This new factorial has Γ-representation

$$y! = \frac{\Gamma(y+1)}{\Gamma(y-[y]+1)}$$

and representation by the Pochhammer symbol (see, e.g., [3]):

$$y! = (y)_{[y]}$$

Obviously, if y is a natural number, $y!$ is the standard function "factorial".

It can be easily seen that the extended function has the properties similar to these of the standard function.

Second, we shall define a new function (it is possible, it is already defined, too, but the author does not know this). It is an inverse function to the function"factorial" and for the arbitraty positive real numbers x and y it has the form:

$$x? = y \text{ \underline{iff} } y! = x. \tag{1}$$

Let us show only one of its integer properties.

For every positive real number x:

$$[(x+1)?] = \begin{cases} [x?]+1, & \text{if there exists a natural number } n \text{ such that} \\ & n! = x+1 \\ [x?], & \text{otherwise} \end{cases}$$

From the above discussion it is clear that we can ignore the new factorial, using the definition

$$x? = y \text{ \underline{iff} } (y)_{[y]} = x.$$

Practically, everywhere below y is a natural number, but in some places x will be a positive real number (but not an integer).

Then the $n-$th member of the first sequence is

$$F_p(n) = [n?]!$$

and of the second sequence it is

$$f_p(n) = \lceil n?\rceil !.$$

The checks of these equalities is direct, or by the method of induction.

Therefore, the values of the n-th partial sums

$$X_n = \sum_{k=1}^{n} F_p(k)$$

and

$$Y_n = \sum_{k=1}^{n} f_p(k)$$

of the two above Smarandache's sequences are, respectively,

$$X_n = \sum_{k=1}^{[n?]} (k! - (k-1)!).(k-1)! + (n - [n?]! + 1).[n?]! \tag{2}$$

and

$$Y_n = \sum_{k=1}^{[n?]} (k! - (k-1)!).k! + (n - [n?]! + 1).[n?]! \qquad (3)$$

The proofs can be made by the induction. For example, the validity of (2) is proved as follows.

Let n = 1. Then the validity of (2) is obvious. Let us assume that (2) is valid for some natural number n. For the form of $n+1$ there are two cases:
(a) for $n+1$ does not exists a natural number m for which $n+1 = m!$. Therefore, $[(n+1)?] = [n?]$ and then

$$X_{n+1} = Y_n + F_p(n+1)$$

$$= \sum_{k=1}^{[n?]} (k! - (k-1)!).(k-1)! + (n - [n?]! + 1).[n?]! + [(n+1)?]!$$

$$= \sum_{k=1}^{[(n+1)?]} (k! - (k-1)!).(k-1)! + ((n+1) - [(n+1)?]! + 1).[(n+1)?]!.$$

(b) for $n+1$ there exists a natural number m for which $n+1 = m!$. Therefore, for $n > 2$ does not exists a natural number m for which $n = m!$, $[(n+1)?] = [n?] + 1$, $[(n+1)?] = n+1$, from (1), and then

$$X_{n+1} = Y_n + F_p(n+1)$$

$$= \sum_{k=1}^{[n?]} (k! - (k-1)!).(k-1)! + (n - [n?]! + 1).[n?]! + [(n+1)?]!$$

$$= \sum_{k=1}^{[n?]} (k! - (k-1)!).(k-1)! + ((n+1) - [n?]!).([(n+1)?] - 1)! + [(n+1)?]!$$

$$= \sum_{k=1}^{[(n+1)?]} (k! - (k-1)!).(k-1)! + ((n+1) - [(n+1)?]! + 1).[(n+1)?]!.$$

Therefore, (2) is valid.
The validity of (3) is proved analogically.

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[2] F. Smarandache, Only problems, not solutions!. Xiquan Publ. House, Chicago, 1993.
[3] L. Comtet, Advanced Combinatorics, D. Reidel Publ. Co., Dordrecht-Holland, 1974.