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ON THE 22-nd, THE 23-th, AND THE 24-th SMARANDACHE'S PROBLEM

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The 22-nd problem from [1] is the following (see also Problem 27 from [2]):
Smarandache square complements:

$$1, 2, 3, 1, 5, 6, 7, 2, 1, 10, 11, 3, 14, 15, 1, 17, 2, 19, 5, 21, 22, 23, 6, 1, 26, 3, 7, 29, 30, 31, 2, 33$$

$$34, 35, 1, 37, 38, 39, 10, 41, 42, 43, 11, 5, 46, 47, 3, 1, 2, 51, 13, 53, 6, 55, 14, 57, 58, 59, 15, 61,$$
$$62, 7, 1, 65, 66, 67, 17, 69, 70, 71, 2, \ldots$$

For each integer n to find the smallest integer k such that nk is a perfect square. (All these numbers are square free.)

The 23-th problem from [1] is the following (see also Problem 28 from [2]):
Smarandache cubic complements:

$$1, 4, 9, 2, 25, 36, 49, 1, 3, 100, 121, 18, 169, 196, 225, 4, 289, 12, 361, 50, 441, 484, 529,$$

$$9, 5, 676, 1, 841, 900, 961, 2, 1089, 1156, 1225, 6, 1369, 1444, 1521, 25, 1681, 1764,$$
$$1849, 242, 75, 2116, 2209, 36, 7, 20, \ldots$$

For each integer n to find the smallest integer k such that nk is a perfect cub. (All these numbers are cube free.)

The 24-th problem from [1] is the following (see also Problem 29 from [2]):
Smarandache $m-$power complements:
For each integer n to find the smallest integer k such that nk is a perfect $m-$power ($m \geq 2$). (All these numbers are $m-$power free.)

Let us define by $c_m(n)$ the $m-$power complement of the natural number n. Let everywhere below $n = \prod_{i=1}^{k} p_i^{a_i}$, where $p_1 < p_2 < \ldots < p_k$ are different prime numbers and $a_1, a_2, \ldots, a_k \geq 1$ are natural numbers.

Every one of the three problems is related to determining of the form of $c_m(n)$ for arbitrary number n. When $m = 2$ we obtain

$$c_2(n) = \prod_{i=1}^{k} p_i^{b_i}.$$

where $b_i \equiv a_i \pmod 2$ and $b_i \in \{0,1\}$ for every $i = 1, 2, ..., k$.

We shall prove that the following properties hold for function c_2

(1) For every natural number n: $n \geq c_2(n)$;

(2) For every natural number n: $n = c_2(n)$ iff $n = \prod_{i=1}^{k} p_i$,

for the different prime numbers $p_1 < p_2 < ... < p_k$;

(3) For every natural number n: $c_2(c_2(n)) = c_2(n)$.

The validity of these assertions is checked easily.

If $n = \prod_{i=1}^{k} p_i$ for the different prime numbers $p_1 < p_2 < ... < p_k$, then, obviously,

$$n = c_2(n).$$

On the other hand, if $n = c_2(n)$, then for every i $(1 \leq i \leq k)$: $a_i = b_i$. But $a_i \geq 1$ and $b_i \leq 1$. Therefore, $a_i = b_i = 1$, i.e., $n = \prod_{i=1}^{k} p_i$.

The check of (3) can be performed as follows. Let

$$c_2(n) = \prod_{i=1}^{k} p_i^{b_i},$$

where $b_i \in \{0,1\}$ for every i $(1 \leq i \leq k)$. Let

$$c_2(c_2(n)) = \prod_{i=1}^{k} p_i^{d_i},$$

where $d_i \in \{0,1\}$ for every i $(1 \leq i \leq k)$.

Now, if for some i $b_i = 0$, then $d_i = 0$, too; and if for some i $b_i = 1$, then $d_i = 1$, too. Therefore,

$$c_2(c_2(n)) = \prod_{i=1}^{k} p_i^{d_i} = \prod_{i=1}^{k} p_i^{b_i} = c_2(n).$$

When $m = 3$ we obtain

$$c_3(n) = \prod_{i=1}^{k} p_i^{b_i},$$

where $b_i \equiv -a_i \pmod 3$ and $b_i \in \{0,1,2\}$ for every $i = 1, 2, ..., k$.

Immediately it can be seen that none of the above three properties is valid for c_3. Now holds the property

(2') For every natural number n:

$$c_3(n) \neq n.$$

Indeed, for the a_1 there are three cases (the same is valid for a_2, ..., a_k, too):

case 1: $a_1 = 3s + 1$ for some integer $s \geq 0$. Then $b_1 = 2$. If $s = 0$, then p_1 is a divisor of n, but p_1^2 is not a divisor of n, while p_1^2 is a divisor of $c_3(n)$; if $s > 0$, then p_1^3 is a divisor of n,

but p_1^3 is not a divisor of $c_3(n)$;
case 2: $a_1 = 3s + 2$ for some integer $s \geq 0$. Then $b_1 = 1$. Therefore p_1^2 is a divisor of n, but p_1^2 is not a divisor of $c_3(n)$;
case 3: $a_1 = 3s$ for some integer $s \geq 1$. Then $b_1 = 0$. Therefore p_1 is a divisor of n, but p_1 is not a divisor of $c_3(n)$.

Finally, when $m \leq 2$ is an arbitrary natural number, then

$$c_m(n) = \prod_{i=1}^{k} p_i^{b_i},$$

where $b_i \equiv -a_i (\text{mod } m)$ and $b_i \in \{0, 1, 2, ..., m-1\}$ for every $i = 1, 2, ..., k$.

If m is an even number, the above property (3) is valid. Property (1) now has the form:
(1") If for every $i = 1, 2, ..., k$ has the form $p_i = [m.s + \frac{m}{2}]$, or $p_i = [m.s + \frac{m}{2} + 1]$, or ... , or $p_i = m.s$, where $[x]$ is the integer part of the real number x, then

$$n \geq c_m(n),$$

but the opposite is not always valid.

Also, in this case the equality (2) has the form:
(2") For every natural number n:

$$n = c_m(n) \text{ \underline{iff} } m = 2s \text{ for some natural number } s \text{ and } n = \prod_{i=1}^{k} p_i^s,$$

for the different prime numbers $p_1 < p_2 < ... < p_k$.

The validity of the following equalities is easily proved:
(4) For every natural number n:

$$n^3 = c_2(n).c_3(n) \text{ \underline{iff} } n = \prod_{i=1}^{k} p_i^s,$$

and
(5) For every three natural numbers n, p, q:

$$c_p(n) = c_q(n) \text{ \underline{iff} for every } i = 1, 2, ..., k: \text{ there exists a natural number } s, \text{ such that } a_i = pqs, \text{ or } a_i = pqs - 1, \text{ or } a_i = pqs - 2, \text{ or } ..., \text{ or } a_i = pqs - min(p, q) + 1.$$

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[1] C. Dumitrescu, V. Seleacu, Some notions and questions in number theory, Erhus Univ. Press, Glendale, 1994.
[2] F. Smarandache, Only problems, not solutions!. Xiquan Publ. House, Chicago, 1993.