NNTDM 5 (1999) 2, 52-54

An inequality concerning the prime numbers

Laurenţiu Panaitopol
Facultatea de Matematică, Universitatea Bucureşti
Str. Academiei nr.14
RO-70109 Bucharest 1, Romania
e-mail address: pan@al.math.unibuc.ro

In this note we shall use the following standard notations: p_n is the n-th prime number, $\pi(x)$ is the number of primes not exceeding x, and $\theta(x) = \sum_{p \le x} \log p$, the sum being taken after all primes p, $p \le x$.

Denoting

$$A_n = \frac{p_1 + p_2 + \ldots + p_n}{n},$$

R. Mandl [1] conjectures that $A_n \le \frac{1}{2}p_n$, for $n \ge 9$. B. Rosser and L. Schoenfeld, in [3], announced the positive answer of this conjecture. In the present note we shall prove a similar result concerning the geometrical mean. Namely, if $G_n = \sqrt[n]{p_1 p_2 \ldots p_n}$, we have the following:

Theorem

$$G_n \le \frac{1}{e}p_n, \text{ for } n \ge 10.$$

Proof: We remark first that $\frac{1}{e}$ is the best possible upper bound for $\frac{G_n}{p_n}$, since it is not hard to prove that $\pi(x)\log(x) - \theta(x) \sim \pi(x)$. We are going to prove the theorem under a modified form, namely

$$\pi(x)(\log x - 1) > \theta(x), \text{ for } x \ge 24.$$

If f is a differentiable function with continuous derivative, then it is well known that

$$\sum_{p \le x} f(p) = f(x)\pi(x) - \int_2^x \pi(y) f'(y) dy.$$

Put $f(x) = \log x$, and it follows that

$$\theta(x) = \log x \pi(x) - \int_2^x \frac{\pi(y)}{y} dy.$$

We are going to use the classical results of Rosser and Schoenfeld [2]:

$$\pi(x) > x(\frac{1}{\log x} + \frac{1}{2\log^2 x}), \text{ for } x \geq 59,$$

and

$$\pi(x) < x(\frac{1}{\log x} + \frac{3}{2\log^2 x}), \text{ for } x > 1.$$

If we denote $I_1 = \int_2^{59} \frac{\pi(y)}{y} dy$ and $I_2 = \int_{59}^x \frac{\pi(y)}{y} dy$, for $x \geq 59$, then we have

$$\pi(x) \log x - \theta(x) = I_1 + I_2.$$

We have

$$I_1 = \sum_{i=1}^{16} \int_{p_i}^{p_i+1} \frac{idy}{y} = \sum_{i=1}^{16} i(\log p_{i+1} - \log p_i) = 16 \log 59 - \sum_{i=1}^{16} \log p_i.$$

After a simple computation we get $I_1 > 20.3$. For I_2 we have

$$I_2 \geq \int_{59}^x (\frac{1}{\log y} + \frac{1}{2\log^2 y}) dy.$$

Successively integrating by parts it follows that

$$I_2 \geq \frac{x}{\log x}(1+\frac{3}{2\log x})+\frac{3x}{\log^3 x} - \frac{59}{\log 59}(1+\frac{3}{2\log 59}+\frac{3}{\log^2 59}) > \pi(x)+\frac{3x}{\log^3 x} - 22.5.$$

From the above it follows that

$$\int_2^x \frac{\pi(y)}{y} dy > \pi(x) + \frac{3x}{\log^3 x} - 2.2.$$

Putting $g(x) = \frac{3x}{\log^3 x} - 2.2$ it follows that $g'(x) = \frac{3(\log x - 3)}{\log^4 x} > 0$ for $x \geq 59$, and since $g(59) > 0.4 > 0$, we get that $g(x) > 0$, so

$$\int_2^x \frac{\pi(y)}{y} dy > \pi(x)$$

for $x \geq 59$. We are going to study this inequality for values less than 59. We denote $h(x) = \int_2^x \frac{\pi(y)}{y} dy - \pi(x)$ and we remark that on an interval I in which there are no primes, the function h is increasing. In order to show that h is positive, it is enough to compute $h(p)$ for p prime, $p < 59$.

We have $h(p_{i+1}) = i \log p_{i+1} - \theta(p_i) - i - 1$. It follows that $h(23) < 0$, $h(24) > 0$, and then $h(p_{i+1}) > 0$ for $9 \leq i \leq 15$, so $h(x) > 0$ for $n \geq 24$, i.e.

$$\pi(x)(\log x - 1) > \theta(x) \text{ for } x \geq 24.$$

For $n \geq 10$, $p_n \geq 24$ and it follows that $n(\log p_n - 1) > \sum_{k=1}^{n} \log p_k$, therefore we get that $\sqrt[n]{p_1 p_2 \dots p_n} < \frac{1}{e} p_n$.

For $4 \leq n \leq 9$ one can show that $\frac{1}{e} p_{n+1} > \sqrt[n]{p_1 p_2 \dots p_n}$, and thus

$$\sqrt[n]{p_1 p_2 \dots p_n} < \frac{1}{e} p_{n+1}, \text{ for } n \geq 4.$$

References

[1] R. Mandl, On the sum and the average of the first primes, Notices Amer. Math. Soc. **21** (1974), A54-A55.

[2] J. B. Rosser, L. Schoenfeld, Approximate formulas for some functions of prime numbers, Illinois J. Math., **6** (1962), 64-89.

[3] J. B. Rosser, L. Schoenfeld, Sharper Bounds for the Chebyshev Functions $\theta(x)$ and $\psi(x)$, Mathematics of Computation, **29**, 129 (1975), 243-269.

AMS Classification Numbers: 11A25, 11N05.