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ON THE 61-st, THE 62-nd, AND THE 63-th SMARANDACHE'S PROBLEM

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The 61-th problem from [1] is the following (see also Problem 66 from [2]):

Smarandache exponents (of power 2):

$$0,1,0,2,0,1,0,3,0,1,0,2,0,1,0,4,0,1,0,2,0,1,0,3,0,1,0,2,0,1,0,5,0,1,0,2,0,1,0,3,$$
$$0,1,0,2,0,1,0,4,0,1,0,2,0,1,0,3,0,1,0,2,0,1,0,6,0,1,...$$

($e_2(n)$ is the largest exponent (of power 2) which divides n.)
Or, $e_2(n) = k$ if 2^k divides n but 2^{k+1} does not.

In [1] and [2] there are two misprints in the above sequence.

The 62-th problem from [2] is the following (see also Problem 67 from [1]):

Smarandache exponents (of power 3):

$$0,0,1,0,0,1,0,0,2,0,0,1,0,0,1,0,0,2,0,0,1,0,0,1,0,0,3,0,0,1,0,0,1,0,0,2,$$
$$0,0,1,0,0,1,0,0,2,0,0,1,0,0,1,0,0,2,0,0,1,0,0,1,0,0,2,0,0,1,0...$$

($e_3(n)$ is the largest exponent (of power 3) which divides n.)
Or, $e_3(n) = k$ if 3^k divides n but 3^{k+1} does not.

The 63-th problem from [2] is the following (see also Problem 68 from [1]):

Smarandache exponents (of power p) { generalization }:
($e_p(n)$ is the largest exponent (of power p) which divides n, where p is an integer ≥ 2.)
Or, $e_p(n) = k$ if p^k divides n but p^{k+1} does not.

Let $[x]$ be the integer part of the real number x.

We can rewrite the first sequence to the form:

$0,1,$
$0,2,0,1,$
$0,3,0,1,0,2,0,1,$
$0,4,0,1,0,2,0,1,0,3,0,1,0,2,0,1,$
$0,5,0,1,0,2,0,1,0,3,0,1,0,2,0,1,0,4,0,1,0,2,0,1,0,3,0,1,0,2,0,1,$
$0,6,0,1,...,$

and we can obtain formulas for the $n-$th member of the new sequence and of the the sum of its first n elements, but the following form of the first sequence is more suitable and the two corresponding formulas will be simpler:

$0,$
$1, 0,$
$2, 0, 1, 0,$
$3, 0, 1, 0, 2, 0, 1, 0,$
$4, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0,$
$5, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 4, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0,$
$6, 0, 1, ...,$

Therefore, the $k-$th row ($k \geq 0$) contains 2^k members and let they be:

$$b_{k,1}, b_{k,2}, ..., b_{k,2^k}$$

and for every $i = 1, 2, ..., 2^{k-1}$:

$$b_{k,2i} = 0.$$

The second form of the sequence shows that for every $k \geq 1$:

$$b_{0,1} = 0,$$

$$b_{k,2i-1} = \begin{cases} k, & \text{if } i = 1 \\ b_{k-1,2i-1}, & \text{if } 2 \leq i \leq 2^{k-2} \\ b_{k-1,2i-2^{k-1}-1}, & \text{if } 2^{k-2}+1 \leq i \leq 2^{k-1} \end{cases} \qquad (1)$$

Obviously, for every two natural numbers k, i there exists a natural number n: $b_{k,i} = e_2(n)$.

Let the natural number n be fixed. Therefore, we can determine the number of the row and the place in this row in which is places $e_2(n)$. They are:

$$k = [log_2 n]$$

and

$$i = n - 2^{[log_2 n]} + 1.$$

Then, from (1) and from the second form of the sequence it follows the following explicit representation:

$$b_{k,2i-1} = \begin{cases} k, & \text{if } i = 1 \\ k-1, & \text{if } i = 2^{k-2}+1 \\ k-2, & \text{if } i = 2^{k-3}+1 \text{ or } i = 2^{k-2}+2^{k-3}+1 \\ k-3, & \text{if } i = 2^{k-4}+1 \text{ or } i = 2^{k-3}+2^{k-4}+1 \\ & \text{or } i = 2^{k-4}+2.2^{k-3}+1 \text{ or } i = 2^{k-4}+3.2^{k-3}+1 \\ k-4, & \text{if } i = 2^{k-5}+1 \text{ or } i = 2^{k-5}+2^{k-4}+1 \\ & \text{or } i = 2^{k-5}+2.2^{k-4}+1 \text{ or } i = 2^{k-5}+3.2^{k-4}+1 \\ & \text{or } i = 2^{k-5}+4.2^{k-4}+1 \text{ or } i = 2^{k-5}+5.2^{k-4}+1 \\ & \text{or } i = 2^{k-5}+6.2^{k-4}+1 \text{ or } i = 2^{k-5}+7.2^{k-4}+1 \\ \vdots & \qquad\qquad \vdots \\ k-s, & \text{if } i = 2^{k-s-1}+1 \text{ or } i = 2^{k-s-1}+2^{k-s}+1 \\ & \text{or } i = 2^{k-s-1}+2.2^{k-s}+1 \text{ or } i = 2^{k-s-1}+3.2^{k-s}+1 \\ & \text{or } \dots \\ & \text{or } i = 2^{k-s-1}+(2^{s-1}-1).2^{k-s}+1 \end{cases}, \qquad (2)$$

for $s < k$.

The validity of (2) is seen direcly by our construction, or it can be proved, e.g., by induction.

Let R_k^2 is the sum of the member from $k-$th row. Easily it can be seen that

$$R_k^2 = 2^k - 1.$$

Now, let S_n^2 be the sum of the first n members of the sequence $\{e_2(n)\}_{n=1}^{\infty}$, i.e.

$$S_n^2 = \sum_{i=1}^{n} e_2(i).$$

From (2) it can be seen that it is valid:

$$S_n^2 = \sum_{j=1}^{[log_2 n]-1} R_j^2 + \sum_{i=1}^{n-2^{[log_2 n]}+1} b_{[log_2 n],i}$$

$$= (2^{[log_2 n]} - 1 - [log_2 n]) + [log_2 n] + \sum_{j=1}^{[log_2 n]} j.[\frac{n - 2^{[log_2 n]} + 2^j}{2^{j+1}}].$$

Therefore,

$$S_n^2 = 2^{[log_2 n]+1} - 1 + \sum_{j=1}^{[log_2 n]} j.[\frac{n - 2^{[log_2 n]} + 2^j}{2^{j+1}}]. \qquad (3)$$

The validity of (3) can be proved, e.g., by induction, using (2).

Also by induction it can be proved that

$$S_n^2 = 2^{[log_2 n]+1} - 1 + \sum_{j=1}^{[log_2 n]} j.([\frac{n - 2^{[log_2 n]}}{2^j}] - [\frac{n - 2^{[log_2 n]} + 2^j}{2^{j+1}}]). \qquad (4)$$

On the other hand, it can be proved directly, that the right parts of (3) and (4) coincide. For this aim it is enough to be proved that for every natural number n and for every natural number j such that $n \geq 2^{j+1}$ the following identity is valid:

$$[\frac{n - 2^{[log_2 n]}}{2^j}] = [\frac{n - 2^{[log_2 n]} + 2^j}{2^{j+1}}] + [\frac{n - 2^{[log_2 n]} + 2^j}{2^{j+1}}],$$

which can be made, e.g., by induction.

Analogically, we shall rewrite the second sequence to the form:

1, 0, 0, 1, 0, 0,
2, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0,
3, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0,
3, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0,
4, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0...

Therefore, the $k-$th row ($k \geq 1$) contains 2.3^k members and let they be:

$$b_{k,1}, b_{k,2}, ..., b_{k,2.3^k}$$

and for every $i = 1, 2, ..., 2.3^{k-1}$:

$$b_{k,3i-1} = b_{k,3i} = 0.$$

The second form of this sequence shows that for every $k \geq 1$:

$$b_{k,3i-2} = \begin{cases} k, & \text{if } i = 1 \\ b_{k-1,3i}, & \text{if } 2 \leq i \leq 2.3^{k-2} \\ b_{k-1,3i-3^{k-1}}, & \text{if } 2.3^{k-2} + 1 \leq i \leq 3^{k-1} \\ b_{k-1,3i-3^k}, & \text{if } 3^{k-1} + 1 \leq i \leq 2.3^{k-1} \end{cases} \tag{5}$$

As in the first case, for every two natural numbers k, i there exists a natural number n: $b_{k,i} = e_2(n)$.

Let the natural number n be fixed. Therefore, we can determine the number of the row and the place in this row in which is places $e_2(n)$. They are:

$$k = [log_3 n]$$

and

$$i = n - 3^{[log_3 n]} + 1.$$

Then, from (5) and from the second form of this sequence it follows the following explicit representation (for $s < k$):

$$
b_{k,3i-2} = \begin{cases}
k, & \text{if } i = 1 \text{ or } i = 3^{k-1}+1 \\
k-1, & \text{if } i = 3^{k-2}+1 \text{ or } i = 2.3^{k-2}+1 \\
 & \text{or } i = 4.3^{k-2}+1 \text{ or } i = 5.3^{k-2}+1 \\
k-2, & \text{if } i = 3^{k-3}+1 \text{ or } i = 2.3^{k-3}+1 \\
 & \text{or } i = 3^{k-3}+3^{k-2}+1 \text{ or } i = 2.3^{k-3}+3^{k-2}+1 \\
 & \text{or } i = 3^{k-3}+2.3^{k-2}+1 \text{ or } i = 2.3^{k-3}+2.3^{k-2}+1 \\
 & \text{or } i = 3^{k-3}+3.3^{k-2}+1 \text{ or } i = 2.3^{k-3}+3.3^{k-2}+1 \\
 & \text{or } i = 3^{k-3}+4.3^{k-2}+1 \text{ or } i = 2.3^{k-3}+4.3^{k-2}+1 \\
 & \text{or } i = 3^{k-3}+5.3^{k-2}+1 \text{ or } i = 2.3^{k-3}+5.3^{k-2}+1 \\
\vdots & \qquad \vdots \\
k-s, & \text{if } i = 3^{k-s-1}+1 \text{ or } i = 2.3^{k-s-1}+1 \\
 & \text{or } i = 3^{k-s-1}+3^{k-s}+1 \text{ or } i = 2.3^{k-s-1}+3^{k-s}+1 \\
 & \text{or } \dots \\
 & \text{or } i = 3^{k-s-1}+(2.3^{s-1}-1).3^{k-s}+1 \\
 & \text{or } i = 2.3^{k-s-1}+(2.3^{s-1}-1).3^{k-s}+1
\end{cases} . \qquad (6)
$$

The validity of (6) is seen direcly by our construction, or it can be proved, e.g., by induction.

Let R_k^3 is the sum of the member from $k-$th row. Easily it can be seen that

$$R_k^3 = 3^k - 1.$$

Now, let S_n^3 be the sum of the first n members of the sequence $\{e_3(n)\}_{n=1}^{\infty}$, i.e.

$$S_n^3 = \sum_{i=1}^{n} e_3(i).$$

From (6) it can be seen that it is valid:

$$S_n^3 = \sum_{j=1}^{[log_3 n]-1} R_j^3 + \sum_{i=1}^{n-3^{[log_3 n]}+1} b_{[log_3 n],i}$$

$$= \frac{3^{[log_3 n]}-1}{2} - [log_3 n] + \sum_{j=1}^{[log_3 n]} j.([\frac{n-3^{[log_3 n]}}{3^{j-1}}] - [\frac{n-3^{[log_3 n]}}{3^j}]) + [log_3 n].$$

Therefore,

$$S_n^3 = \frac{3^{[log_3 n]}-1}{2} + \sum_{j=1}^{[log_3 n]} j.([\frac{n-3^{[log_3 n]}}{3^j}] - [\frac{n-3^{[log_3 n]}}{3^{j+1}}]). \qquad (7)$$

The validity of (7) can be proved, e.g., by induction, using (6).

By analogy with the above constructions, we can write the sequence of the $p-$th powers, where p is a prime number in the form:

$$\underbrace{\underbrace{0,\dots,0}_{p-1},1,\underbrace{0,\dots,0}_{p-1},1}_{p-1}$$

$$\underbrace{\underbrace{0,\dots,0}_{p-1},2,\underbrace{\underbrace{0,\dots,0}_{p-1},1,\dots,\underbrace{0,\dots,0}_{p-1},1}_{p-1},\dots,\underbrace{0,\dots,0}_{p-1},2,\underbrace{\underbrace{0,\dots,0}_{p-1},1,\dots,\underbrace{0,\dots,0}_{p-1},1}_{p-1}}_{p-1}$$

$$\underbrace{0,\dots,0}_{p-1},3,\underbrace{\underbrace{0,\dots,0}_{p-1},1,\dots,\underbrace{0,\dots,0}_{p-1},1}_{p-1},\underbrace{\underbrace{0,\dots,0}_{p-1},2,\underbrace{\underbrace{0,\dots,0}_{p-1},1,\dots,\underbrace{0,\dots,0}_{p-1},1}_{p-1},\dots,\underbrace{0,\dots,0}_{p-1},2,\underbrace{\underbrace{0,\dots,0}_{p-1},1,\dots,\underbrace{0,\dots,0}_{p-1},1}_{p-1}}_{p-1}$$

$$\underbrace{0,\dots,0}_{p-1},3,\underbrace{\underbrace{0,\dots,0}_{p-1},1,\dots,\underbrace{0,\dots,0}_{p-1},1}_{p-1},\underbrace{\underbrace{0,\dots,0}_{p-1},2,\underbrace{\underbrace{0,\dots,0}_{p-1},1,\dots,\underbrace{0,\dots,0}_{p-1},1}_{p-1},\dots,\underbrace{0,\dots,0}_{p-1},2,\underbrace{\underbrace{0,\dots,0}_{p-1},1,\dots,\underbrace{0,\dots,0}_{p-1},1}_{p-1}}_{p-1}$$

$$\bullet \qquad \bullet \qquad \bullet$$

$$\underbrace{0,\dots,0}_{p-1},3,\underbrace{\underbrace{0,\dots,0}_{p-1},1,\dots,\underbrace{0,\dots,0}_{p-1},1}_{p-1},\underbrace{\underbrace{0,\dots,0}_{p-1},2,\underbrace{\underbrace{0,\dots,0}_{p-1},1,\dots,\underbrace{0,\dots,0}_{p-1},1}_{p-1},\dots,\underbrace{0,\dots,0}_{p-1},2,\underbrace{\underbrace{0,\dots,0}_{p-1},1,\dots,\underbrace{0,\dots,}_{p-1}1}_{p-1}}_{p-1}$$

Therefore, the $k-$th row ($k \geq 1$) contains $(p-1).p^k$ members and let they be:

$$b_{k,1}, b_{k,2}, ..., b_{k,(p-1).p^k}$$

and for every $i = 1, 2, ..., (p-1).p^{k-1}$:

$$b_{k,p.i-p+2} = b_{k,p.i-p+3} = ... = b_{k,p.i} = 0.$$

The second form of this sequence shows that for every $k \geq 1$:

$$b_{k,p.i-p+1} = \begin{cases} k, & \text{if } i = 1 \\ b_{k-1,p.i}, & \text{if } 2 \leq i \leq (p-1).p^{k-2} \\ b_{k-1,p.i-p^{k-2}}, & \text{if } (p-1).p^{k-2}+1 \leq i \leq p^{k-2} \\ \vdots & \vdots \\ b_{k-1,p.i-s.p^{k-1}}, & \text{if } s.p^{k-1}+1 \leq i \leq (s+1).p^{k-1} \\ & \text{for } s = 1, 2, ..., p-2 \end{cases} \tag{8}$$

As in the first case, for every two natural numbers k, i there exists a natural number n: $b_{k,i} = e_p(n)$.

Let the natural number n be fixed. Therefore, we can determine the number of the row and the place in this row in which is places $e_p(n)$. They are:

$$k = [log_p n]$$

and

$$i = n - p^{[log_p n]} + 1.$$

Then, from (8) and from the second form of this sequence it follows the following explicit representation:

$$b_{k,p.i-p+1} = \begin{cases} k, & \text{if } i = 1 \text{ or } i = p^{k-1}+1 \\ & \text{or } i = 2.p^{k-1} \text{ or } i = (p-2).p^{k-1}+1 \\ k-1, & \text{if } i = p^{k-2}+1 \text{ or } i = 2.p^{k-2}+1 \\ & \text{or ... or } i = (p-1).p^{k-2}+1 \\ & \text{or } i = (p+1).p^{k-2}+1 \text{ or ...} \\ & \text{or } i = (2p-1).p^{k-2}+1 \text{ or ...} \\ & \text{or } i = ((p-2).p)+1).p^{k-2}+1 \text{ or ...} \\ & \text{or } i = ((p-1).p)-1).p^{k-2}+1 \\ \vdots & \vdots \\ k-s, & \text{if } i = p^{k-s-1}+1 \text{ or } i = 2.p^{k-s-1}+1 \\ & \text{or ... or } i = (p-1).p^{k-s-1}+1 \\ & \text{or } i = p^{k-s-1}+p^{k-s}+1 \text{ or ... or } i = (p-1).p^{k-s-1}+p^{k-s}+1 \text{ ...} \\ & \text{or } i = p^{k-s-1}+((p-1).p^{s-1}-1).p^{k-s}+1 \text{ ...} \\ & \text{or } i = (p-1).p^{k-s-1}+((p-1).p^{s-1}-1).p^{k-s}+1 \end{cases}, \tag{9}$$

for $s < k$.

The validity of (9) is seen direcly by our construction, or it can be proved, e.g., by induction.

Let R_k^p is the sum of the member from $k-$th row. Easily it can be seen that

$$R_k^p = p^k - 1.$$

Now, let S_n^p be the sum of the first n members of the sequence $\{e_p(n)\}_{n=1}^{\infty}$, i.e.

$$S_n^p = \sum_{i=1}^{n} e_p(i).$$

From (9) it can be seen that it is valid:

$$S_n^p = \sum_{j=1}^{[log_p n]-1} R_j^p + \sum_{i=1}^{n-p^{[log_p n]}+p} b_{[log_p n],i}$$

$$= \frac{p^{[log_p n]+1} - 1}{p-1} + \sum_{j=1}^{[log_p n]} j.([\frac{n - p^{[log_p n]}}{p^j}] - [\frac{n - p^{[log_p n]}}{p^{j+1}}]). \quad (10)$$

The validity of (10) can be proved, e.g., by induction, using (9).

Finally, we shall note that (10) can be used for representation of $n!$. It is (see [3])

$$n! = \prod_{p \in \mathcal{P}} (\frac{p^{[log_p n]+1} - 1}{p-1} + \sum_{j=1}^{[log_p n]} j.([\frac{n - p^{[log_p n]}}{p^j}] - [\frac{n - p^{[log_p n]}}{p^{j+1}}])).$$

or

$$n! = \prod_{i=1}^{\pi(n)} (\frac{p^{[log_p n]+1} - 1}{p-1} + \sum_{j=1}^{[log_p n]} j.([\frac{n - p^{[log_p n]}}{p^j}] - [\frac{n - p^{[log_p n]}}{p^{j+1}}])),$$

where

$$\mathcal{P} = \{p_1, p_2, p_3, ...\} = \{2, 3, 5, ...\}$$

is the set of the prime numbers.

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