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DIOPHANTINE QUADRUPLES AND QUINTUPLES MODULO 4

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Abstract: A Diophantine m-tuple with the property $D(n)$ is a set $\{a_1, a_2, \dots a_m\}$ of positive integers such that for $1 \le i < j \le m$, the number $a_i a_j + n$ is a perfect square. In the present paper we give necessary conditions that the elements a_i of a set $\{a_1, a_2, a_3, a_4, a_5\}$ must satisfy modulo 4 in order to be a Diophantine quintuple.

Let n be an integer. A set of positive integers $\{a_1, a_2, \dots, a_m\}$ is called *a Diophantine m-tuple with the property* $D(n)$, or P_n*-set of size* m, if $a_i a_j + n$ is a perfect square for all $1 \le i < j \le m$. A P_n-set X will be termed *extendable* if, for some integer d, $d \notin X$, the set $X \cup \{d\}$ is a P_n-set.

The problem of extending P_n-sets is an old one, dating from the time of Diophantus (see [4, 5]). The first P_1-set of size 4 was found by Fermat, and it was $\{1, 3, 8, 120\}$. The most famous result on P_n-sets is due to Baker and Davenport [2], who proved that if $\{1, 3, 8, d\}$ is a P_1-set, then d has to be 120.

In 1985, Brown [3], Gupta and Singh [8] and Mohanty and Ramasamy [9] proved independently that if $n \equiv 2 \pmod 4$, then there does not exist a P_n-set of size 4. In 1993, the author proved that if $n \not\equiv 2 \pmod 4$ and $n \notin \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there exists at least one P_n-set of size 4 (see [6]). P_n-sets of size 5 were studied in [1, 7, 10].

The purpose of the present paper is to characterize congruence types modulo 4 of Diophantine quadruples and quintuples. We will say that a set $X = \{a_1, \dots, a_m\}$ has *a congruence type* $[b_1, \dots, b_m]$, where $b_i \in \{0, 1, 2, 3\}$, if $a_i \equiv b_i \pmod 4$ for $i = 1, \dots, m$.

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Our starting point is the following result of Mootha and Berzsenyi [11, Theorems 1, 2 and 3].

Theorem 1 **(a)** *If all of the elements of a P_n-set of size $m \geq 3$ are odd, then they are congruent to one another, modulo 4.*

(b) *If only one of the elements of P_n-set of size $m \geq 3$ is odd, then all of the others are congruent to 0, modulo 4.*

(c) *P_n-sets of the congruence type $[1, 2, 3]$ are not extendable.*

Proof: **(a)** Let $\{a, b, c\}$ be a P_n-set. Assume that a, b, c are odd and $a \equiv b \equiv c - 2 \pmod 4$. Since square of an integer is congruent to 0 or 1 modulo 4, $ab + n = \square$ implies $n \equiv 0, 3 \pmod 4$, and $ac + n = \square$ implies $n \equiv 1, 2 \pmod 4$. Contradiction.

(b) Assume that $\{a, b, c\}$ is a P_n-set, a is odd, b is even and $c \equiv 2 \pmod 4$. Then $ac + n = \square$ implies $n \equiv 2, 3 \pmod 4$, and $bc + n = \square$ implies $n \equiv 0, 1 \pmod 4$. Contradiction.

(c) Assume that $\{a, b, c, d\}$ is a P_n-set, $a \equiv 1 \pmod 4$, $b \equiv 2 \pmod 4$ and $c \equiv 3 \pmod 4$. Applying **(a)** on the set $\{a, c, d\}$ we see that d cannot be odd, and applying **(b)** on the set $\{a, b, d\}$ we see that d cannot be even. ∎

Theorem 2 *A P_n-set of size 4 has one of the following congruence types:*

$$[0,0,0,0], \quad [0,0,0,2], \quad [0,0,2,2], \quad [0,2,2,2], \quad [2,2,2,2],$$

$$[0,0,0,1], \quad [0,0,0,3], \quad [0,0,1,1], \quad [0,0,1,3], \quad [0,0,3,3],$$

$$[0,1,1,1], \quad [0,3,3,3], \quad [2,1,1,1], \quad [2,3,3,3], \quad [1,1,1,1], \quad [3,3,3,3],$$

and all of these congruence types are indeed possible.

Proof: The first part of the theorem follows directly from Theorem 1, and the second part will follow from Theorem 4 below. ∎

Theorem 3 *A P_n-set of size 5 has one of the following congruence types:*

$$[0,0,0,0,0], \quad [0,0,0,0,2], \quad [0,0,0,2,2], \quad [0,0,2,2,2], \quad [0,2,2,2,2],$$

$$[2,2,2,2,2], \quad [0,0,0,0,1], \quad [0,0,0,0,3], \quad [0,0,0,1,1], \quad [0,0,0,1,3],$$

$$[0,0,0,3,3], \quad [0,0,1,1,1], \quad [0,0,3,3,3], \quad [0,1,1,1,1], \quad [2,1,1,1,1],$$

$$[0,3,3,3,3], \quad [2,3,3,3,3], \quad [1,1,1,1,1], \quad [3,3,3,3,3].$$

Proof: The theorem is a direct consequence of Theorem 2. ∎

Theorem 4 *For all congruence types from Theorem 3, apart from maybe $[1,1,1,1,1]$ and $[3,3,3,3,3]$, there exists a nonzero integer n and a P_n-set of size 5 with that congruence type.*

Proof: The theorem follows from the following table:

n	P_n-set of size 5	Congruence type
-1196	$\{28,44,60,84,180\}$	$[0,0,0,0,0]$
-455	$\{8,72,102,148,492\}$	$[0,0,0,0,2]$
1600	$\{8,42,250,768,22272\}$	$[0,0,0,2,2]$
1024	$\{2,66,210,640,36480\}$	$[0,0,2,2,2]$
14400	$\{26,200,266,506,9450\}$	$[0,2,2,2,2]$
-299	$\{14,22,30,42,90\}$	$[2,2,2,2,2]$
1024	$\{4,33,2660,5520,245760\}$	$[0,0,0,0,1]$
9216	$\{12,99,7980,16560,737280\}$	$[0,0,0,0,3]$
400	$\{4,21,125,384,11136\}$	$[0,0,0,1,1]$
-255	$\{8,32,77,203,528\}$	$[0,0,0,1,3]$
-476	$\{20,31,75,96,192\}$	$[0,0,0,3,3]$
400	$\{4,21,69,125,384\}$	$[0,0,1,1,1]$
400	$\{7,12,63,128,375\}$	$[0,0,3,3,3]$
3600	$\{13,100,133,253,4725\}$	$[0,1,1,1,1]$
-3185325	$\{1113,2958,3417,3993,4725\}$	$[2,1,1,1,1]$
1296	$\{11,35,128,243,315\}$	$[0,3,3,3,3]$
-353925	$\{371,986,1139,1331,1575\}$	$[2,3,3,3,3]$

■

Corollary 1 *For all congruence types from Theorem 3, there exists an integer n and a P_n-set of size 5 with that congruence type.*

Proof: The statement follows directly from Theorem 4, using the fact that $\{1,9,25,49,81\}$ and $\{3,27,75,147,243\}$ are P_0-sets. ■

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