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SOME OBSERVATIONS ON THE DIOPHANTINE EQUATION $u^2 + v^2 = x^3 + y^3$

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1. Introduction

The sums of pairs of integers go back to the Pythagoreans and the sums of pairs of cubes go back to Diophantos at least, though perhaps their most famous expression was when the dying Ramanujan observed that Hardy's taxicab number, 1729, was "a very interesting number. It is the smallest number expressible as the sum of two cubes in two different ways" (Hardy, 1967: 37):

$$1729 = 12^3 + 1^3 = 10^3 + 9^3.$$

It is the purpose of this note to comment on some properties of sums of two squares which are equal to the sums of two cubes, a digression begun in an earlier note (Clarke, Shannon and Leyendekkers, in press).

2. Observations

If a solution in integers exists for the pair of Diophantine equations

$$u^m + v^m = r^n, \tag{1}$$

and

$$x^n + y^n = s^m, \tag{2}$$

then these solutions may be used to solve in integers

$$u^m + v^m = x^n + y^n. \tag{3}$$

Let a solution of (1) be denoted by (u_1, v_1, r_1) and solutions of (2) by (x_1, y_1, s_1). As an example, consider

$$2^2 + 11^2 = 5^3 \text{ and } 1^3 + 2^3 = 3^2,$$

so that $(u_1, v_1, r_1) = (2, 11, 5)$ and $(x_1, y_1, s_1) = (1, 2, 3)$ with $m = 2, n = 3$. For all such solutions

$$(u_1^m + v_1^m)(x_1^n + y_1^n) = (u_1^m + v_1^m)x_1^n + (u_1^m + v_1^m)y_1^n$$

$$= r_1^n x_1^n + r_1^n y_1^n$$

$$= h_1^n + k_1^n$$

where $h_1 = r_1 x_1, k_1 = r_1 y_1$. Similarly,

$$\begin{aligned}(x_1^n + y_1^n)(u_1^m + v_1^m) &= s_1^m u_1^m + s_1^m v_1^m \\ &= p_1^m + q_1^m\end{aligned}$$

where $p_1 = s_1 u_1, q_1 = s_1 v_1$. Thus

$$p_1^m + q_1^m = h_1^n + k_1^n.$$

If solutions to (1) and (2) are known, then the solution to (3) can be found, but numerical examples show that the converse is not true. Solutions to

$$u^2 + v^2 = x^3 + y^3 \tag{4}$$

are thus

$$(3 \times 2)^2 + (3 \times 11)^2 = (5 \times 1)^3 + (5 \times 2)^3,$$

that is, $6^2 + 33^2 = 5^3 + 10^3$. It will be noted that if n is odd, then there is a factor, $x + y$, so that $u^m + v^m$ and $x^n + y^n$ must be composite. A solution to

$$x^2 + y^2 = z^3 \tag{5}$$

is $x = a(a^2 - 3), y = 3a^2 - 1, z = a^2 + 1, \ a > 3$ (Burton, 1980: 257).

Other solutions exist: for example, let $x = 2a^3, y = 2a^3$ in $x^2 + y^2 = z^3$, then

$$(2a^3)^2 + (2a^3)^2 = (2a^2)^3.$$

Similarly for

$$x^3 + y^3 = z^2, \tag{6}$$

let $x = 2a^2, y = 2a^2$, then $(2a^2)^3 + (2a^2)^3 = (4a^3)^2$.

This can be extended to Equations (1) and (2). However, it is likely that solutions other than these exist; for instance, for $x^2 + y^2 = z^3$, $x = a(a^2 - 3)$, $y = 3a^2 - 1$, $z = a^2 + 1$ and $x = 2a^3, y = 2a^3, z = 2a^2$.

Note that when $a = 2$, Equation (5) has a solution $8^2 + 8^2 = 32^2 = s^2$, and Equation (6) has a solution $16^2 + 16^2 = 8^3 = r^3$ From these we can deduce a solution to Equation (3) when $m = 2, n = 3$,

$$\begin{aligned}p_1 &= s_1 u_1, \ \therefore p_1 = 32 \times 16 = 512, \\ q_1 &= s_1 v_1, \ \therefore q_1 = 32 \times 16 = 512,\end{aligned}$$

$$h_1 = r_1 x_1, \qquad \therefore h_1 = 8 \times 8 = 64,$$
$$k_1 = r_1 y_1, \qquad \therefore k_1 = 8 \times 8 = 64.$$

Thus $512^2 + 512^2 = 64^3 + 64^3$, which is somewhat trivial and if of a type discussed in Clarke *et al* (in press).

3. Conclusion

In the foregoing, the integers in the sums of two cubes are not relatively prime, nor are the constituent elements of the sums of two cubes. A topic for further study is to impose relatively prime conditions. Nowak (1997), for instance, has extended the work of Moroz(1985) on primitive lattice points; in particular, he has sharpened the estimate of the average order of the arithmetic function $\rho_3^+(n)$ (as well as $\rho_3^-(n)$) in which $\rho_3^+(n)$ counts the number of ways to write a natural number n as the sum of two cubes of relatively prime integers:

$$\rho_3^+(n) = \#\{(u,v) \in \mathbb{N}^2 : u^k + v^k = n, gcd(uv) = 1\}$$

which is generated by the Dirichlet series of $Z_k^+(s)/4\zeta(ks)$, where $\zeta(.)$ represents the Riemann zeta function and

$$Z_k^+(s) \;=\; \sum_{(u,v)\in \mathbf{Z}^2 : |u|^k + |v|^k > 0} (|u|^k + |v|^k), \; (\Re s > 1).$$

The average order for the sum of relatively prime squares, $\rho_2^+(n)$ has been studied by Baker (1994).

References

Baker, R.C. 1994. The square-free divisor problem. *The Quarterly Journal of Mathematics. Oxford.* **45**:269-277.

Burton, David M. 1980. *Elementary Number Theory*. Boston: Allyn and Bacon.

Clarke, J.H., Shannon, A.G. & Leyendekkers, J.V. In press. On sums of pairs of squares and cubes. *Notes on Number Theory & Discrete Mathematics*.

Hardy, G.H. 1967. *A Mathematician's Apology*. (With a foreword by C.P. Snow.) Cambridge: Cambridge University Press.

Moroz, B.Z. 1985. On the number of primitive lattice points in plane domains. *Monatschefte für Mathematik*. **99**: 37-43.

Nowak, Werner Georg. 1997. On sums and differences of two relative prime cubes, II. *Tatra Mountains Mathematical Publications*. **11**: 23-34.

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