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SOME REPRESENTATIONS RELATED TO $n!$

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Devoted to Prof. John Turner
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In the present paper we describe different (well-known and new) representations of $n!$ for the natural number n and comment them. The following denotations are used everywhere below: $[x]$ is the integer part of the real nonnegative number x; $ord_p m$ is the largest exponent of power p which divides the natural number m, where p is a prime number, i.e.,$ord_p m = k$ if and only if p^k divides m, but p^{k+1} does not (another notation for $ord_p m$ is $e_p(m)$); Λ marks Mangoldt's function; μ marks Möbius function; $\mathcal{P}$ is the set of all primes; $S_n^{(m)}$ denotes Stirling numbers of first kind; $\left(\begin{array}{c} n \\ k \end{array}\right)$ is used for binomial coefficients.

First, we give a short list of formulas for $ord_p n!$:

$$ord_p n! = \sum_{k=1}^{[log_p n]} [\frac{n}{p^k}]; \tag{1}$$

$$ord_p n! = \sum_{k=1}^{[\frac{n}{p}]} [log_p \frac{n}{k}]; \tag{2}$$

$$ord_p n! = - \sum_{m=1}^{[\frac{n}{p}]} \sum_{k=1}^{[\frac{n}{mp}]} \mu(kp)[\frac{n}{kmp}]; \tag{3}$$

$$ord_p n! = 2ord_p[\frac{n}{2}]! + \frac{1}{2}[log_p n] - \frac{1}{2} \sum_{k=1}^{[log_p n]} (-1)^{[\frac{n}{p^k}]}, \tag{4}$$

and esspecially for $ord_2 n!$:

$$ord_2 n! = \frac{1}{2} \sum_{k=1}^{[log_2 n]} (-1)^{[\frac{n}{2^k}]} - \frac{1}{2}[log_2 n] + 2[\frac{n}{2}]; \tag{5}$$

$$ord_2 n! = \sum_{k=1}^{[log_2 n]} k.[\frac{n+2^k}{2^{k+1}}]. \tag{6}$$

Identity (1) is known as Legendre's formula for $ord_p n!$. Identity (2) holds from (1) and from the identity

$$\sum_{k=1}^{[log_p n]} [\frac{n}{p^k}] = \sum_{k=1}^{[\frac{n}{p}]} [log_p \frac{n}{k}]. \tag{7}$$

Identity (7) is a particular case from the following identity proved in [1]:

$$\sum_{k=1}^{[\Psi^{-1}(n)]} [\frac{n}{\Psi(k)}] = \sum_{k=1}^{[\frac{n}{\Psi(1)}]} [\Psi^{-1}(\frac{n}{k})], \tag{8}$$

under the assumption that $\Psi(x)$ is an increasing function on $[1, \infty)$ with $\Psi(1) > 0$, where Ψ^{-1} is the inverse function of Ψ.

Putting in (8) $\Psi(m) = p^m$ and using that $\Psi^{-1}(s) = log_p s$, we get (7).

Identity (3) follows from the relation

$$-\sum_{k=1}^{[\frac{n}{p}]} \mu(kp)[\frac{n}{kp}] = [log_p n], \tag{9}$$

proved in [2], and from (2).

We need the representation $(-1)^{[x]} = 1 - 2[x] + 4[\frac{x}{2}]$, which is valid for any real $x \geq 0$, in order to prove (4) and after that to use (1). (5) is a corollary from (4) when $p = 2$, while formulas (6) and (9) are proposed by Ivan Mladenov (our fellow-student since our being in the Mathematical Faculty of the Sofia University). The proof of (6) is based on the equality

$$[\frac{x}{2^k}] - [\frac{x}{2^{k+1}}] = [\frac{x + 2^k}{2^{k+1}}],$$

which is valid for any real $x \geq 1$.

Second, we give a short list of formulas for $n!$. Using the obvious formula

$$n! = \prod_{p \in \mathcal{P}, p \leq n} p^{ord_p n!} = \prod_{p \in \mathcal{P}} p^{ord_p n!}, \tag{10}$$

i.e., the product is finite, because for $p > n : prd_p n! = 0$, we may use each one of formulas (1), (2) and (3) to represent $n!$. Other representations are given below.

$$n! = \sum_{m=0}^{n} (-1)^{n-m} S_n^{(m)}; \tag{11}$$

$$n! = exp(\sum_{k=1}^{n} \Psi(\frac{n}{k})), \tag{12}$$

where Ψ denotes Chebishev's function, i.e., $\Psi(x) = \sum_{n=1}^{[x]} \Lambda(n)$;

$$n! = exp(\sum_{k=1}^{n} \Lambda(k).[\frac{n}{k}]); \tag{13}$$

$$n! = [\frac{k^n}{\binom{n}{k}}]), \text{ where } k \geq 2.n^{n+2}; \quad (14)$$

$$n! = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x-k)^n, \quad (15)$$

for every complex number x;

$$n! = \prod_{p\in\mathcal{P}, p\leq n} (\frac{p^{[log_p n]+1}-1}{p-1} + \sum_{j=1}^{[log_p n]} j.([\frac{n-p^{[log_p n]}}{p^j}] - [\frac{n-p^{[log_p n]}}{p^{j+1}}]). \quad (16)$$

Identities (11) and (12) are well-known (see, e.g., [3,4]). For the proof of (13) it is necessary firstly to note that $\sum_{d|k} \Lambda(d) = ln\ k$, where d runs all divisors of k. Therefore,

$$\sum_{k=1}^{n} \sum_{d|k} \Lambda(d) = \sum_{k=1}^{n} ln\ k = ln\ n!. \quad (17)$$

Using the well-known identity $\sum_{k=1}^{n} f(k) \sum_{d|k} g(d) = \sum_{k=1}^{n} g(k) \sum_{s=1}^{[\frac{n}{k}]} f(ks)$ in the case $f \equiv 1$ and $g \equiv \Lambda$, we obtain

$$\sum_{k=1}^{n} \sum_{d|k} \Lambda(d) = \sum_{k=1}^{n} \Lambda(k).[\frac{n}{k}]. \quad (18)$$

Then (17) and (18) yield (13). Identity (14) is contained in [5], and identity (15) is proved in [6], while the proof of the particular case $x = 0$ is given in [7].

Identity (16) can be proved as by induction, as well as directly from the construction from [8], where the 66-th, 67-th, and 68-th Smarandache's problems (see [9,10]) are solved. We would like to note that in [9,10] there are other problems, related to the function "factorial", too.

Another representation of $n!$ is based on (10), (4) and the following form of the Chebishev's function (see [11]):

$$\Psi(n) = \sum_{p\in\mathcal{P}} [log_p n].lnp = ln \prod_{p\in\mathcal{P}} p^{[log_p n]},$$

i.e.,

$$e^{\Psi(n)} = \prod_{p\in\mathcal{P}} p^{[log_p n]}.$$

It has the form:

$$n! = \prod_{p\in\mathcal{P}} p^{ord_p n!} = \prod_{p\in\mathcal{P}} p^{2ord_p[\frac{n}{2}]!+\frac{1}{2}[log_p n]-\frac{1}{2}\sum_{k=1}^{[log_p n]}(-1)^{[\frac{n}{p^k}]}}$$

$$= \frac{\prod\limits_{p\in\mathcal{P}} (p^{ord_p[\frac{n}{2}]!})^2 . \sqrt{\prod\limits_{p\in\mathcal{P}} p^{[log_p n]}}}{\sqrt{\prod\limits_{p\in\mathcal{P}} p^{\sum\limits_{k=1}^{[log_p n]} (-1)^{[\frac{n}{p^k}]}}}}$$

$$= \frac{\prod\limits_{p\in\mathcal{P}} ([\frac{n}{2}]!)^2 . \sqrt{e^{\Psi(n)}}}{\sqrt{\prod\limits_{p\in\mathcal{P}} p^{\sum\limits_{k=1}^{[log_p n]} (-1)^{[\frac{n}{p^k}]}}}},$$

i.e., we received the formula

$$n! = \frac{\prod\limits_{p\in\mathcal{P}} ([\frac{n}{2}]!)^2 . \sqrt{e^{\Psi(n)}}}{\sqrt{\prod\limits_{p\in\mathcal{P}} p^{\sum\limits_{k=1}^{[log_p n]} (-1)^{[\frac{n}{p^k}]}}}},$$

which admits an interesting interpretation.

Namelly, it is known that $e^{\Psi(n)}$ is the Least Common Multiple LCM of the natural numbers smollar than n. Therefore,

$$n! = \frac{\prod\limits_{p\in\mathcal{P}} ([\frac{n}{2}]!)^2 . \sqrt{LCM(1,2,...,n)}}{\sqrt{\prod\limits_{p\in\mathcal{P}} p^{\sum\limits_{k=1}^{[log_p n]} (-1)^{[\frac{n}{p^k}]}}}}.$$

Another form of the same formula is the following:

$$LCM(1,2,...,n) = \frac{(n!)^2}{([\frac{n}{2}]!)^4} . \prod_{p\in\mathcal{P}} p^{\sum\limits_{k=1}^{[log_p n]} (-1)^{[\frac{n}{p^k}]}}.$$

Of course, the product in the right hand side of each of the above three formulas is finite and it is restricted up to $p \leq n$, because, if $p > n$, then $[log_p n] = 0$ and the sum $\sum\limits_{k=1}^{[log_p n]}$ vanishes.

We shall finish with a new result related to the concept "factorial". The concepts of $n!!$ is already introduced and there are some problems in [9,10] related to it. Let us define the new factorial $n!!!$ only for numbers with the forms $3k+1$ and $3k+2$ ($k \geq 0$):

$$n!!! = 1.2.4.5.7.8.10.11.....n$$

For $n = \sum_{i=1}^{m} a_i.10^{m-i} \equiv \overline{a_1 a_2 ... a_m}$, where a_i is a natural number and $0 \leq a_i \leq 9$ ($1 \leq i \leq m$) let (see [12]):

$$\varphi(n) = \begin{cases} 0 & , \text{ if } n = 0 \\ \sum_{i=1}^{m} a_i & , \text{ otherwise} \end{cases}$$

and for the sequence of functions $\varphi_0, \varphi_1, \varphi_2, ...$, such that $\varphi_0(n) = n$, $\varphi_{k+1} = \varphi(\varphi_k(n))$, where ($k$ is a natural number), let the function ψ be defined by $\psi(n) = \varphi_l(n)$, where l is the smallest natural number for which $\varphi_{l+1}(n) = \varphi_l(n)$.

Let the sequence $a_1, a_2, ...$ with member - natural numbers, be given and let $c_i = \psi(a_i)$ ($i = 1, 2, ...$). Hence we deduce the sequence $c_1, c_2, ...$ from the former sequence. If k and l exists, such that $l \geq 0, c_{i+l} = c_{k+i+l} = c_{2k+i+l} = ...$ for $1 \leq i \leq k$, then we shall say that $< c_{l+1}, c_{l+2}, ..., c_{l+k} >$ is a base of the sequence $c_1, c_2, ...$ with a length k and with respect to function ψ.

An obvious, but unpulbished up to now result is that the sequence $\{\psi(n!)\}_{n=1}^{\infty}$ has a base with a length of 1 with respect to the function ψ and it is $< 9 >$. The first members of this sequence are $1, 2, 6, 6, 3, 9, 9, 9, ...$

We shall prove that the sequence $\{\psi(n!!!)\}_{n=1}^{\infty}$ has a base with a length of 12 with respect to the function ψ and it is $< 1, 2, 8, 4, 1, 8, 8, 7, 1, 5, 8, 1 >$.

Really, the validity of the assertion for the first 12 natural numbers with the above mentioned forms, i.e., the numbers $1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17$, is checked directly. Let us assume that the assertion is valid for the numbers

$$(18k+1)!!!, (18k+2)!!!, (18k+4)!!!, (18k+5)!!!, (18k+7)!!!, (18k+8)!!!, (18k+10)!!!,$$

$$(18k+11)!!!, (18k+13)!!!, (18k+14)!!!, (18k+16)!!!, (18k+17)!!!.$$

Then

$$\psi((18k+19)!!!) = \psi((18k+17)!!!.(18k+19)) = \psi(\psi(18k+17)!!!.\psi(18k+19)) = \psi(1.1) = 1;$$

$$\psi((18k+20)!!!) = \psi((18k+19)!!!.(18k+20)) = \psi(\psi(18k+19)!!!.\psi(18k+20)) = \psi(1.2) = 2;$$

$$\psi((18k+22)!!!) = \psi((18k+20)!!!.(18k+22)) = \psi(\psi(18k+20)!!!.\psi(18k+22)) = \psi(2.4) = 8,$$

etc., with which the assertion is proved.

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