First-order recurrence relations for the Chebyshev polynomials and associated functions

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Abstract

The Chebyshev polynomials of the first kind, $T_n(x) = \cos(n\cos^{-1} x)$ (n integer, $|x| \le 1$), satisfy the second-order recurrence relation

$$T_{n+2} = 2xT_{n+1} - T_n, \quad T_0 = 1, \quad T_1 = x.$$

It is shown that they also satisfy the first-order recurrence relation

$$T_{n+1} = xT_n + \mathrm{r}((1-x^2)(1-T_n^2)), \quad T_0 = 1,$$

where the function r is defined by

$$\mathrm{r}(p(x)^2) = \mathrm{slc}(p(x))p(x)$$

for polynomial $p(x)$ and $\mathrm{slc}(p(x))$ denotes the sign of the leading coefficient of $p(x)$.

Associated Chebyshev polynomials, satisfying

$$X_{n+2} = 2a(x)X_{n+1} - X_n, \quad X_0 = x_0, \quad X_1 \text{ polynomial},$$

for polynomial $a(x)$, are then defined and the corresponding first-order relation given. An example of non-polynomial $a(x)$ leading to the functions $V_n(x) = \sin(n\sin^{-1} x)$ is also mentioned together with a more general first-order recurrence relation for the non-polynomial case.

1. Chebyshev polynomials of the first kind

Consider the first-order recurrence relation

$$T_{n+1}^2 - 2xT_nT_{n+1} + T_n^2 = 1 - x^2, \quad T_0 = 1, \tag{1}$$

motivated by the trigonometric expansion of $T_{n+1}(x) = \cos((n+1)\cos^{-1} x)$. The second-order relations associated with (1) can be determined as follows.

Equation (1) implies that

$$(T_{n+1} - xT_n)^2 = (1-x^2)(1-T_n^2) \tag{2}$$

and, by symmetry,

$$(T_n - xT_{n+1})^2 = (1-x^2)(1-T_{n+1}^2). \tag{3}$$

Incrementing the subscripts of (2) and comparing this with (3) gives

$$(T_{n+2} - xT_{n+1})^2 = (T_n - xT_{n+1})^2 \quad \Rightarrow \quad T_{n+2} - xT_{n+1} = \pm(T_n - xT_{n+1})$$

so that

$$T_{n+2} = 2xT_{n+1} - T_n \quad \text{or} \tag{4}$$

$$T_{n+2} = T_n. \tag{5}$$

Also, substitution of $T_0 = 1$ into (1) gives

$$T_1^2 - 2xT_1 + x^2 = 0 \quad \Rightarrow \quad (T_1 - x)^2 = 0 \quad \Rightarrow \quad T_1 = x.$$

Equation (4) with the given initial conditions produces the Chebyshev polynomials of the first kind (1, x, $2x^2 - 1$, $4x^3 - 3x$, ...), while equation (5) gives the trivial sequence 1, x, 1, x, ... (This spurious solution was introduced by the squaring process involved in obtaining equation (1) from the trigonometric expansion of $T_{n+1}(x)$.)

In order to solve the quadratic (1) for T_{n+1} in terms of T_n, the usual quadratic formula is not appropriate if polynomial solutions are required. This is because, for a polynomial $p(x)$, $\sqrt{p(x)^2} = |p(x)|$ is not in general a polynomial. To avoid this difficulty the following function is introduced.

For a polynomial $p(x)$, let

$$\mathrm{r}(p(x)^2) = \mathrm{slc}(p(x))p(x)$$

where $\mathrm{slc}(p(x))$ indicates the sign of the leading coefficient of $p(x)$. We note that $\mathrm{r}(p(x)^2)$ is then a polynomial with positive leading coefficient and that $\mathrm{r}(c)$ reduces to the usual square root function for constant $c \geq 0$ ($\mathrm{r}(c) = r((\pm\sqrt{c})^2) = \sqrt{c}$).

Thus, from (2),

$$T_{n+1} = xT_n \pm \mathrm{r}((1-x^2)(1-T_n^2)) \tag{6}$$

is an alternative first-order form which also leads to equations (4) and (5) but which maintains the polynomial character of the solutions. (The definition of the function r applies here since (2) implies that if T_n and T_{n+1} are polynomials then $(1-x^2)(1-T_n^2) = p(x)^2$ for some polynomial $p(x)$.)

To determine when solutions to equations (4) or (5) will arise, consider the case of equation (6) with positive sign. Suppose $\deg(T_n) = q$ and T_n has positive leading coefficient. The RHS of (6) gives

$$\deg(T_{n+1}) = \max(q+1, (2q+2)/2) = q+1.$$

The function r maintains the polynomial nature of the solutions and, in this case, the degree of the polynomial increases at each iteration. Thus, the Chebyshev polynomials, given by the usual second-order relation (4), are also given by the first order relation

$$T_{n+1} = xT_n + \mathrm{r}((1-x^2)(1-T_n^2)). \tag{7}$$

Example 1

Equation (7) indicates an alternative, first-order, process for determining the Chebyshev polynomials. With $T_0 = 1$, equation (7) gives

$$T_1 = x + \mathrm{r}(0) = x,$$
$$T_2 = x^2 + \mathrm{r}((1-x^2)^2) = 2x^2 - 1, \ldots .$$

There are in fact (infinitely many) other non-polynomial functions which satisfy (1), such as the usual quadratic formula solution

$$T_{n+1} = xT_n \pm \sqrt{(1-x^2)(1-T_n^2)},$$

but these are merely point-wise combinations of solutions to (4) and (5) and are not polynomial in general. For example, $T_{n+1} = xT_n + \sqrt{(1-x^2)(1-T_n^2)}$ with $T_0 = 1$ gives

$$T_1 = x \quad \text{and}$$

$$T_2 = x^2 + |1-x^2| = \begin{cases} 2x^2 - 1 & |x| \geq 1, \\ 1 & |x| < 1. \end{cases}$$

Continued iteration of this solution intermingles the basic polynomial solutions to (4) and (5) since the sign of the last term of the recurrence relation does not always serve to increase the degree of the polynomial over the whole range of x values.

2. Associated Chebyshev functions

The simple approach demonstrated in the foregoing is readily applied to a related recurrence relation which determines polynomials associated with the Chebyshev polynomials (the main difference being that restrictions on T_0 and T_1 are relaxed). We generalize (1) by setting

$$X_{n+1}^2 - 2a(x)X_nX_{n+1} + X_n^2 = x_0^2 - b(x), \quad X_0 = x_0, \tag{8}$$

where $a(x)$ and $b(x)$ are polynomials and x_0 is constant. Following the steps used to obtain equations (2)-(5) gives

$$X_{n+2} = 2a(x)X_{n+1} - X_n \quad \text{or} \tag{9}$$

$$X_{n+2} = X_n.$$

Substitution of $X_0 = x_0$ into (8) leads to

$$X_1 = a(x)x_0 \pm r(a(x)^2x_0^2 - b(x))$$

when $a(x)^2x_0^2 - b(x)$ is a perfect square. By choosing x_0 and $b(x)$ suitably, a range of initial conditions can be obtained. Exactly analogous to (6), polynomial solutions are given by

$$X_{n+1} = a(x)X_n \pm \mathrm{r}(x_0^2 - b(x) - (1-a(x)^2)X_n^2), \quad X_0 = x_0. \tag{10}$$

If $\mathrm{slc}(a(x)) < 0$ and solutions with increasing degree are required, then the choice of sign in (10) may alternate (depending on the degree of $b(x)$).

Example 2

Consider the case where $\mathrm{slc}(a(x)) > 0$ and polynomials of increasing degree are required.

When $X_0 = x_0 = 1$ and $b(x) \equiv a(x)^2$ we have $X_1 = a(x)$ which, with (9), indicates that $X_n = T_n(a(x))$. Equation (10) gives an alternative, first-order, method of calculating these polynomials by use of $X_{n+1} = a(x)X_n + \mathrm{r}((1-a(x)^2)(1-X_n^2))$.

When $X_0 = x_0 = 1$ and $b(x) \equiv 0$ we obtain $X_1 = 2a(x)$. It is well known that equation (9) with these initial conditions gives the Chebyshev polynomials of the second kind, ie $X_n = U_n(a(x))$, where $U_n(x) = \frac{1}{n+1}T'_{n+1}(x)$. Equation (10) thus also gives an alternative method of calculating these polynomials, $X_{n+1} = a(x)X_n + \mathrm{r}(1-(1-a(x)^2)X_n^2)$.

Suitable choices of $a(x)$ then give the shifted Chebyshev polynomials $T_n^*(x) = T_n(2x-1)$, $U_n^*(x) = U_n(2x-1)$, $C_n(x) = 2T_n(x/2)$ and $S_n(x) = U_n(x/2)$ mentioned by Abramowitz and Stegun [1]. The Morgan-Voyce and associated polynomials [2], which arise from a similar second-order recurrence relation, may also be mentioned as special cases. Each of these polynomials can thus also be calculated by first-order formulae.

3. Non-polynomial $a(x)$

Finally, it is interesting to note that first-order recurrence relations can be found for non-polynomial functions such as $V_n = \sin(n\sin^{-1}x)$ (n integer, $|x| \le 1$). V_n satisfies

$$T_n^2 + V_n^2 = 1 \qquad n \text{ even},$$

$$V_n = (-1)^{(n-1)/2}T_n \qquad n \text{ odd}.$$

It can be verified by substitution and use of the trigonometric properties that V_n satisfies the associated Chebyshev recurrence relation (9) with a non-polynomial function $a(x)$, i.e.

$$V_{n+2} = 2\sqrt{1-x^2}V_{n+1} - V_n, \quad V_0 = 0, \quad V_1 = x.$$

An alternative, first-order form can be found by expansion of $V_{n+1}(x) = \sin((n+1)\sin^{-1} x)$ and use of $\cos(n\sin^{-1} x) = \cos(n\cos^{-1}\sqrt{1-x^2}) = T_n(\sqrt{1-x^2})$, viz.,

$$V_{n+1}(x) = \sqrt{1-x^2}V_n(x) + xT_n(\sqrt{1-x^2}), \quad V_0 = 0.$$

This suggests consideration of first-order relations of the form

$$X_{n+1} = a(x)X_n + xT_n(a(x)), \quad X_0 = x_0, \tag{11}$$

where $a(x)$ is not necessarily polynomial. For this relation we have

$$\begin{aligned} X_{n+2} - 2a(x)X_{n+1} + X_n &= a(x)X_{n+1} + xT_{n+1}(a(x)) \\ &\quad - 2a(x)\big(a(x)X_n + xT_n(a(x))\big) \\ &\quad + a(x)X_{n-1} + xT_{n-1}(a(x)) \\ &= a(x)\big(X_{n+1} - 2a(x)X_n + X_{n-1}\big) \end{aligned}$$

by use of the recurrence relation for T_{n+1}. Thus, X_n $(n > 2)$ also satisfies the second-order associated Chebyshev relation (9) whenever X_0, X_1 and X_2 do. Substituting X_0, X_1 and X_2, found by use of (11), into equation (9) forces either $x_0 = 0$ or $a(x) \equiv \pm 1$ in this case.

Conclusion

A range of well-known functions commonly described by second-order recurrence relations in fact satisfy a generalized first-order relation, providing alternative methods for their calculation and analysis. Are other well-known recurrence relations, such as the Fibonacci relation $F_{n+2} = F_{n+1} + F_n$ ($F_0 = 0$, $F_1 = 1$), inherently second-order or do equivalent first-order relations also exist?

References

1. Abramowitz, M. and Stegun, I.A., 1970. Handbook of mathematical functions. Dover Publications, NY.
2. Horadam, A.F., 1998. New aspects of Morgan-Voyce polynomials. In: Applications of Fibonacci numbers, Vol 7, pp.161-176. G.E. Bergum et al (eds), Klewer Academic Publishers, Netherlands.