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ON SUMS OF PAIRS OF SQUARES AND CUBES

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1. Introduction

The purpose of this note is to outline some results for sums of pairs of squares and cubes of integers, and then to pose a relevant question. The history of such questions goes back to Legendre (Dickson, 1965, 261) who established a description of the natural numbers which are the sums of 3 squares, and to Euler (Dickson, 1952, 230-231) who answered the case for 2 squares. More recently, Ewell (1983) showed that an easy special case of the triple-product identity implies Fermat's two-square theorem: any prime of the form $4k+1$ can be expressed as the sum of two squares.

2. Sums of Squares

Let $n = x^2 + y^2$, $x, y \in \mathbb{Z}$, then a well-known theorem in number theory (Burton, 1980) states that $n = N^2 m$, $N \in \mathbb{Z}$, m is square-free and has no factors of the form $4k+3$. The value of k may be zero or a positive integer. N may be equal to unity. Let $z = a + bi$ where $a, b \in \mathbb{Z}$, $i^2 = -1$. Then

$$\begin{aligned} z^p &= (a+bi)^p, \;\; p \in \mathbb{Z}_+, \\ &= a^p + \binom{p}{1} a^{p-1} bi - \binom{p}{2} a^{p-2} b^2 - \binom{p}{3} a^{p-3} b^3 i + \ldots \\ &= [a^p - \binom{p}{2} a^{p-2} b^2 + \binom{p}{4} a^{p-4} b^4 - \ldots] + \\ &\quad [\binom{p}{1} a^{p-1} b - \binom{p}{3} a^{p-3} b^3 + \binom{p}{5} a^{p-5} b^5 - \ldots] i. \end{aligned}$$

Both series in the square brackets terminate when $r \geq p$. We then take the complex conjugates, $z\bar{z}$ and $z^p \bar{z}^p$, and use induction to show that

$$(a^2+b^2)^p = [a^p - \binom{p}{2} a^{p-2} b^2 + \ldots]^2 + [\binom{p}{1} a^{p-1} b - \binom{p}{3} a^{p-3} b^3 + \ldots]^2 \quad (2.1)$$

For example, when $p = 5$,

$$\begin{aligned} z^p \bar{z}^p &= (a^5 - 10a^3b^2 + 5ab^4)^2 + (5a^4b - 10a^2b^3 + b^5)^2 \\ &= a^{10} + 5a^8b^2 + 10a^6b^4 + 10a^4b^6 + 5a^2b^8 + b^{10} \end{aligned}$$

$$= (a^2 + b^2)^5. \tag{2.2}$$

As a numerical illustration, let $a = 3$ and $b = 2$:

$$13 = N^2 m = 1^2(4 \times 3 + 1) = 3^2 + 2^2.$$

From (2.2):

$$597^2 + 122^2 = (3^2 + 2^2)^5 = 371293.$$

Since n may not be uniquely expressed as the sum of two squares, there may sometimes be another pair of squares whose sum is equal to the pth power. For instance,

$$145 = 1^2(4 \times 36 + 1) = 8^2 + 9^2 = 12^2 + 1^2.$$

Conversely, the pth root of some integers may be expressed as the sum of the squares of integers from (2.1).

3. Sums of Cubes

Consider the Diophantine equation

$$c^3 = a^3 + b^3. \tag{3.1}$$

Wiles and Taylor (van der Poorten, 1996) have established Fermat's Last Theorem. Here we consider (3.1) as a lead into the last section. Suppose

$$c = a + b + m, \quad m \in \mathbb{Z}. \tag{3.2}$$

Then

$$c^3 = a^3 + b^3 + 3ab(a+b) + m^3 + 3m^2(a+b) + 3(a+b)^2 m.$$

If (3.2) holds, then

$$m^3 + 3(a+b)m^2 + 3(a+b)^2 m + 3ab(a+b) = 0.$$

This is a cubic in m with roots $\alpha, \beta, \gamma,$ say. Then

$$-3(a+b) = \alpha + \beta + \gamma,$$

and

$$3(a+b)^2 = \alpha\beta + \beta\gamma + \gamma\alpha.$$

Thus

$$(\alpha + \beta + \gamma)^2 = 3(\alpha\beta + \beta\gamma + \gamma\alpha),$$

and so

$$\alpha^2 + \beta^2 + \gamma^2 = \alpha\beta + \beta\gamma + \gamma\alpha,$$

which means

$$\alpha = \beta = \gamma$$

and

$$\alpha = m = -(a + b).$$

Substitution into (3.2) yields

$$c = a + b - (a + b) = 0,$$

so that there are no non-trivial solutions of (3.1) (cf. Hunter, 1964). Leyendekkers *et al* (1997) have generalised this in a modular ring. For the general theory of symmetric functions the reader is referred to Macmahon (1915). In the context of this note the issue is to what extent this approach could be applied to the question in the final section.

4. Sums of Squares and Cubes

There is an infinity of integer solutions of

$$u^2 + v^2 = x^3 + y^3. \tag{4.1}$$

This may be verified by substitution

$$u = a^{3m}s^3, v = b^{3n}t^3, x = a^{2m}s^2, y = b^{2n}t^2, \ a, b, m, n, s, t \in \mathbb{Z}.$$

For example, when $a = 2, b = 3, m = 1, n = 4, s = t = 1$,

$$u = 2^3, v = 3^{12}, x = 2^2, y = 3^8.$$

Thus,

$$8^2 + 531441^2 = 4^3 + 6561^3. \tag{4.2}$$

While the parameters s and t merely ensure that (4.1) gives an infinity of solutions, there are other solutions too. For instance,

$$\begin{aligned} 22^2 + 6^2 &= 2^3 + 8^3, \\ 3^2 + 19^2 &= 3^3 + 7^3, \\ 3^2 + 28^2 &= 4^3 + 9^3. \end{aligned}$$

5. Conclusion

The particular case of equation (4.1) can be re-written in terms of the Fibonacci and Lucas numbers (Hoggatt, 1969) as

$$F_3^2 + (F_4^6)^2 = (L_3^2)^3 + (L_2^4)^3.$$

In itself this is neither here nor there, but it raises the question to what extent can Equation (4.1) be generalised in terms of the Fibonacci and/or Lucas numbers by analogy with the way Horadam (1961) generalised the Pythagorean equation:

$$(F_n F_{n+3})^2 + (2F_{n+1}F_{n+2})^2 = (2F_{n+1}F_{n+2} + F_n^2)^2,$$

though Horadam actually proved the result for generalised Fibonacci numbers, and it was generalised even further by Shannon and Horadam (1973). Possible avenues of pursuit are the results:

$$F_{3(n+2)} = 4F_{3(n+1)} + F_{3n}, \qquad \text{(Shannon and Horadam, 1979)},$$

$$F_{n+1}^2 + F_n^2 = F_{2n+1}, \qquad \text{(Hoggatt, 1969)},$$

$$F_{n+1}^3 + F_n^3 = F_{3n} + F_{n-1}^3 \qquad \text{(Vorob'ev, 1961)}.$$

We note in conclusion for the interested reader that Melham (submitted) has generalised the last two results and stated a pertinent conjecture.

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