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ON SOME ARITHMETIC SETS

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This paper is an continuation of our paper [1].

The arithmetic function ∂ is introduced in [2] for every natural number $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, where for $i = 1, 2, ..., k \geq 1$: p_i are prime numbers and $\alpha_i \geq 1$ and it has the following form:

$$\partial(n) = \sum_{i=1}^{k} \alpha_i p_1^{\alpha_1} ... p_{i-1}^{\alpha_{i-1}} p_i^{\alpha_i - 1} p_{i+1}^{\alpha_{i+1}} p_k^{\alpha_k}. \tag{1}$$

Easily it can be seen from (1) that

$$\partial(n) = n \sum_{i=1}^{k} \frac{\alpha_i}{p_i}. \tag{2}$$

From (1) and (2) we see also that for every prime number p:

$$\partial(p) = 1 \tag{3}$$

and

$$\partial(n) \geq n \text{ iff } \sum_{i=1}^{k} \frac{\alpha_i}{p_i} \geq 1.$$

Let

$$C_k = \{x \mid [\frac{\partial(x)}{x}] = k\}.$$

THEOREM 1: For every natural number $k \geq 0$:
(a) $C_k \neq \emptyset$,
(b) $\underline{card} C_k = \aleph_0$.

Proof: From (3) it is clear that for every prime number p:

$$[\frac{\partial(p)}{p}] = 0.$$

Let us assume that for the natural number k there is a natural number n such that:

$$[\frac{\partial(n)}{n}] = k.$$

Let $p \notin \underline{set}(n)$, where for the above natural number n $\underline{set}(n) = \{p_1, p_2, ..., p_k\}$.
Let us construct the natural number $m = np^p$. Then from (2)

$$[\frac{\partial(m)}{m}] = [\sum_{i=1}^{k} \frac{\alpha_i}{p_i} + \frac{p}{p}] = [\sum_{i=1}^{k} \frac{\alpha_i}{p_i}] + 1 = k + 1.$$

Therefore, for the natural number $k+1$ also there is a natural number m such that $m \in C_{k+1}$.

For every natural number n:

$$\underline{card}(\underline{set}(n)) < \aleph_0,$$

where as it is well known $\underline{card}(X)$ is the cardinality of the set X and $\aleph_0$ is the cardinality of the set of the natural numbers. Therefore

$$\underline{card}(\mathcal{P} - \underline{set}(n)) = \aleph_0,$$

where $\mathcal{P}$ is the set of all prime numbers.

Therefore, there is an infinite number of prime pumbers, which can be used in the above construction at the place of the number p. Hence, for every natural number k: $\underline{card}(C_k) = \aleph_0$.

Following [3], we can formulate and prove the following

THEOREM 2: Let $f(m)$ be one of the following expressions:

$$\frac{\psi(m)}{\varphi(m)}, \frac{\sigma(m)}{\varphi(m)}, \frac{\sigma^2(m)}{\varphi(m)}, \frac{\psi(m)}{m}, \frac{m}{\varphi(m)}, \frac{\sigma(m)}{m}, \frac{\Phi(m)}{\varphi^2(m)}, \frac{\psi(m).\psi(m)}{Phi(m)},$$

$$\frac{\sigma(m)}{\varphi(m)}. \frac{\sigma(m)}{\psi(m)}, \frac{\Phi(m)}{m.\varphi(m)}, \frac{m.\psi(m)}{\Phi(m)}, \frac{\psi^2(m)}{\Phi(m)}.$$

For every natural number a the set

$$F_f(a) = \{x \mid (x \in \mathcal{N})\&([f(x)] = a)\},$$

has infinitely many elements x for which $\mu(x) \neq 0$, where μ is the Möbius function, where $\mathcal{N}$ is the set of the natural numbers.

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The authors thank to Prof. ...for his review and for his remark in "Mathematical Reviews" on our paper [1]. really, everywhere in this paper the expression $\mu(x) = 0$ must be read $\mu(x) \neq 0$, because of one and the same misprint. The same correction is necessary for all corresponding places in the paper [3], too. The correct form of Theorem 1 from [3] is the following

Let $\{p_t\}_{t=1}^{\infty}$ be an increasing sequence of primes and $\{\theta_t\}_{t=1}^{\infty}$ satisfies the conditions:

- For every $t \in \mathcal{N}$ we have $\theta_t \in (1, \frac{1+\sqrt{5}}{2})$;
- For every $t \in \mathcal{N}$ it is fulfiled

$$\frac{1}{\theta_{t+1}-1} - \frac{1}{\theta_t - 1} \geq 1$$

- The sequence $\{a_n\}_{n=1}^{\infty}$ converges to $+\infty$, where for $n \in \mathcal{N}$

$$a_n = \theta_1.\theta_2 \ldots \theta_n.$$

If a multiplicative function f satisfies the relations

$$f(p_t) = \theta_t, \; t \in \mathcal{N},$$

then for every $a \in \mathcal{N}$ the set $F_f(a)$ has infinitely many elements x, for which it is fulfiled

$$\mu(x) \neq 0,$$

where μ is the classical Möbius function.

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