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ON THE 40-th AND THE 41-th SMARANDACHE'S PROBLEMS

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In [1] Florian Smarandache formulated 105 unsolved problems.

The 40-th problem is the following (see also Problems 39 and 40 from [2]):

(Inferior) square part:

$$0,1,1,1,4,4,4,4,4,9,9,9,9,9,9,9,16,16,16,16,16,16,16,16,16,25,25,25,25,25,25,25,$$
$$25,25,25,25,36,36,36,36,36,36,36,36,36,36,36,36,36,49,49,\ldots$$

(the largest square less than or equal to n.)
(Superior) square part:

$$0,1,4,4,4,9,9,9,9,9,16,16,16,16,16,16,16,25,25,25,25,25,25,25,25,25,36,36,$$
$$36,36,36,36,36,36,36,36,36,36,36,49,49,\ldots$$

(the smallest square greater than or equal to n.)
Study these sequences.

The 41-th problem is the following (see also Problems 41 and 42 from [2]):

(Inferior) square part:

$$0,1,1,1,1,1,1,1,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,27,27,27,27,27,27,27,27,27,$$
$$27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,$$
$$27,64,64,64,\ldots$$

(the largest cube less than or equal to n.)

$$0,1,8,8,8,8,8,8,8,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,64,64,$$
$$64,64,64,64,64,64,64,64,64,64,64,64,64,64,64,64,64,64,64,64,64,64,64,64,64,64,64,$$
$$64,64,64,64,64,64,64,64,64,64,64,125,125,125,\ldots$$

(the smallest cube greater than or equal to n.)
Study these sequences.

Below use shall use the usual notations: $[x]$ and $\lceil x\rceil$ for the integer part of the real number x and for the least integer $\geq x$, respectively.

The authors think that this is one of the most trivial Smatrandache's problems. The n-th term of every one of the above sequences is, respectively

$$a_n = [\sqrt{n}]^2,$$

of the second -

$$b_n = \lceil\sqrt{n}\rceil^2,$$

of the third -

$$c_n = [\sqrt[3]{n}]^3,$$

and of the fourth -

$$d_n = \lceil\sqrt[3]{n}\rceil^3.$$

The checks of these equalities is direct, or by the induction.

Easily we can prove the validity of the following equalities:

$$\sum_{k=1}^{n} (2k-1).k^2 = \frac{n(n+1)(3n^2+n-1)}{6}, \quad (1)$$

$$\sum_{k=1}^{n} (2k+1).k^2 = \frac{n(n+1)(3n^2+5n+1)}{6}, \quad (2)$$

$$\sum_{k=1}^{n} (3k^2-3k+1).k^3 = \frac{n(n+1)(5n^4+4n^3-4n^2-n+1)}{10}, \quad (3)$$

$$\sum_{k=1}^{n} (3k^2+3k+1).k^3 = \frac{n(n+1)(5n^4+16n^3+14n^2+5n+1)}{10}. \quad (4)$$

For example

$$\sum_{k=1}^{n} (2k-1).k^2 = 2.\sum_{k=1}^{n} k^3 - \sum_{k=1}^{n} k^2$$

$$= 2.\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)(3n^2+n+1)}{6},$$

i.e. (1) is true.

Now using (1) - (4), we shall show the values of the n-th partial sums $A_n = \sum_{k=1}^{n} a_k$, $B_n = \sum_{k=1}^{n} b_k$, $C_n = \sum_{k=1}^{n} c_k$ and $D_n = \sum_{k=1}^{n} d_k$, of the four Smarandache's sequences. They are, respectively,

$$A_n = \frac{[\sqrt{n}-1]([\sqrt{n}-1]+1)(3[\sqrt{n}-1]^2+5[\sqrt{n}-1]+1)}{6} + (n-[\sqrt{n}]^2+1).[\sqrt{n}]^2, \quad (5)$$

$$B_n = \frac{[\sqrt{n}]([\sqrt{n}]+1)(3[\sqrt{n}]^2+[\sqrt{n}]-1)}{6} + (n-[\sqrt{n}]^2).\lceil\sqrt{n}\rceil^2, \tag{6}$$

$$C_n = \frac{[\sqrt[3]{n}-1]([\sqrt[3]{n}-1]+1)(5[\sqrt[3]{n}-1]^4+16[\sqrt[3]{n}-1]^3+14[\sqrt[3]{n}-1]^2+[\sqrt[3]{n}-1]-1)}{10}$$
$$+(n-[\sqrt[3]{n}]^3+1).[\sqrt[3]{n}]^3, \tag{7}$$

$$D_n = \frac{[\sqrt[3]{n}]([\sqrt[3]{n}]+1)(5[\sqrt[3]{n}]^4+4[\sqrt[3]{n}]^3-4[\sqrt[3]{n}]^2-[\sqrt[3]{n}]+1)}{10} + (n-[\sqrt[3]{n}]^3+1).\lceil\sqrt[3]{n}\rceil^3. \tag{8}$$

The proofs can be made again by the induction. For example, the validity of (6) is proved as follows.

Let n = 1. Then the validity of (6) is obvious. Let us assume that (6) is valid for some natural number n. For the form of n there are three cases:
(a) n and $n+1$ are not squares. Therefore, $[\sqrt{n+1}] = [\sqrt{n}]$ and $\lceil\sqrt{n+1}\rceil = \lceil\sqrt{n}\rceil$ and then

$$B_{n+1} = B_n + b_{n+1} = B_n + \lceil\sqrt{n+1}\rceil^2 = B_n + \lceil\sqrt{n}\rceil^2$$
$$= \frac{[\sqrt{n}]([\sqrt{n}]+1)(3[\sqrt{n}]^2+[\sqrt{n}]-1)}{6} + (n-[\sqrt{n}]^2).\lceil\sqrt{n}\rceil^2 + \lceil\sqrt{n}\rceil^2$$
$$= \frac{[\sqrt{n+1}]([\sqrt{n+1}]+1)(3[\sqrt{n+1}]^2+[\sqrt{n+1}]-1)}{6} + (n+1-[\sqrt{n+1}]^2).\lceil\sqrt{n+1}\rceil^2;$$

(b) n is a square (hence, $n+1$ is not a square). Therefore, $[\sqrt{n+1}] = [\sqrt{n}]$, $n = [\sqrt{n}]^2 = [\sqrt{n+1}]^2$ and $\lceil\sqrt{n+1}\rceil = \lceil\sqrt{n}\rceil + 1$ and then

$$B_{n+1} = B_n + b_{n+1} = B_n + \lceil\sqrt{n+1}\rceil^2$$
$$= \frac{[\sqrt{n}]([\sqrt{n}]+1)(3[\sqrt{n}]^2+[\sqrt{n}]-1)}{6} + (n-[\sqrt{n}]^2).\lceil\sqrt{n}\rceil^2 + \lceil\sqrt{n+1}\rceil^2$$
$$= \frac{[\sqrt{n}]([\sqrt{n}]+1)(3[\sqrt{n}]^2+[\sqrt{n}]-1)}{6} + 0 + 1.\lceil\sqrt{n+1}\rceil^2$$
$$= \frac{[\sqrt{n+1}]([\sqrt{n+1}]+1)(3[\sqrt{n+1}]^2+[\sqrt{n+1}]-1)}{6} + (n+1-[\sqrt{n+1}]^2).\lceil\sqrt{n+1}\rceil^2;$$

(c) $n+1$ is a square (for $n > 1$ it follows that n is not a square). Therefore, $[\sqrt{n+1}] = [\sqrt{n}]+1$ and $\lceil\sqrt{n+1}\rceil = \lceil\sqrt{n}\rceil$ and then

$$B_{n+1} = B_n + b_{n+1} = B_n + \lceil\sqrt{n+1}\rceil^2 = B_n + \lceil\sqrt{n}\rceil^2$$
$$= \frac{[\sqrt{n}]([\sqrt{n}]+1)(3[\sqrt{n}]^2+[\sqrt{n}]-1)}{6} + (n-[\sqrt{n}]^2).\lceil\sqrt{n}\rceil^2 + \lceil\sqrt{n}\rceil^2.$$

From the equalities

$$n+1 = [\sqrt{n+1}]^2 = ([\sqrt{n}]+1)^2,$$
$$\lceil\sqrt{n}\rceil = [\sqrt{n}]+1$$

and

$$(n-[\sqrt{n}]^2+1).\lceil\sqrt{n}\rceil^2 = (([\sqrt{n}]+1)^2-[\sqrt{n}]^2).\lceil\sqrt{n}\rceil^2 = (2[\sqrt{n}]+1)([\sqrt{n}]+1)^2$$

it follows that

$$B_{n+1} = \frac{[\sqrt{n}]([\sqrt{n}]+1)(3[\sqrt{n}]^2+[\sqrt{n}]-1)}{6} + (2[\sqrt{n}]+1)([\sqrt{n}]+1)^2$$

$$= \frac{([\sqrt{n}]+1)([\sqrt{n}]+2)(3([\sqrt{n}]+1)^2+[\sqrt{n}])}{6}$$

$$= \frac{[\sqrt{n+1}]([\sqrt{n+1}]+1)(3([\sqrt{n+1}]^2+[\sqrt{n+1}]-1)}{6} + (n+1-[\sqrt{n+1}]^2).[\sqrt{n+1}].$$

Therefore, (6) is valid.

The validity of formulas (5), (7) and (8) are proved analogically.

REFERENCE:

[1] F. Smarandache, Only problems, not solutions!. Xiquan Publ. House, Chicago, 1993.

[2] C. Dumitrescu, V. Seleacu, Some notions and questions in number theory, Erhus Univ. Press, Glendale, 1994.