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HALF- AND ONE-THIRD-REGULAR HEXAGONS

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Devoted to Prof. Anthony Shannon
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As it is well known, the figure numbers (see e.g., [1]) have representations in the form of fugures constructed by mutually touching circles of diameter 1. For example, the second standard hexagonal number has the representation from Fig. 1.

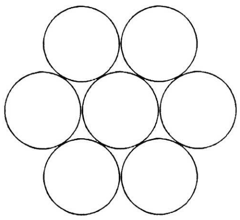

Fig. 1.

Here we shall discuss two modifications of the hexagonal numbers.

1. By "a Half-Regular Hexagon" (HRH) $ABCDEF$ we will understand such hexagon that the lenghts of its sides satisfy the following requirements: $|AB| = |CD| = |EF|$ and $|BC| = |DE| = |FA|$ and all of its angles equal to the regular hexagon angles of $120°$.

The HRH with lengths of its sides k and l will be called a (k,l)-HRH. For example, if $k = 2$ and $l = 3$ the (k,l)-HRH has the representation shown on Fig. 2.

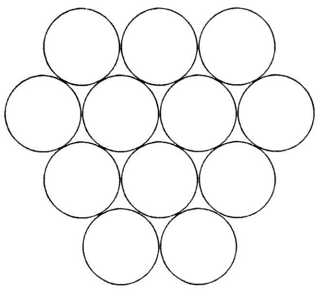

Fig. 2.

Obviously, the $(k,1)$-HRH and $(1,k)$-HRH coincide with the k-th triangular number for every natural number k and also (k,k)-HRH coincides with the regular hexagon with side length of k.

The number $P(k,l)$ of the circles in the (k,l)-HRH corresponds to the (k,l)-th HRH number. The following theorem explains this correspondence:

THEOREM 1: For every two natural numbers k and l:

$$P(k,l) = \frac{(k+l)(k+l-3)}{2} + kl + 1. \tag{1}$$

Proof: We shall use the mathematical induction. Let the natural number k be fixed. If $l = 1$, then

$$P(k,1) = \frac{k(k+1)}{2} = \frac{(k+1)(k-2)}{2} + k + 1 = \frac{(k+1)(k+1-3)}{2} + 1.k + 1,$$

i.e., (1) is valid.

Let us assume that (1) be valid for some fixed natural number l. Now we shall prove that (1) is valid for $l+1$, too.

Immediatelly we can see that the $(k,l+1)$-th HRH can be represented by the (k,l)-th HRH as it is shown on Fig. 3.

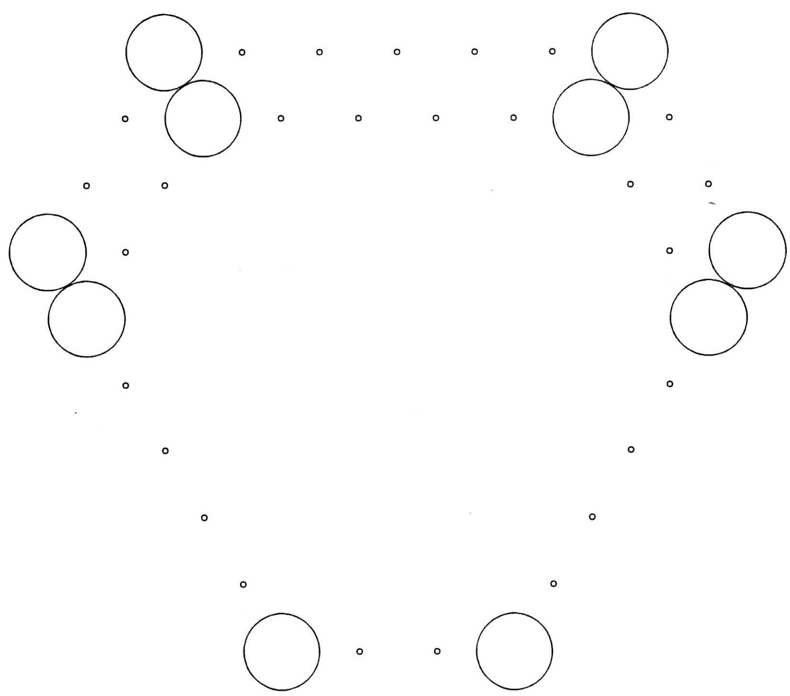

Fig. 3.

Therefore, we can write that

$$P(k,l+1) = P(k,l) + 2k + l - 1 = \frac{(k+l)(k+l-3)}{2} + kl + 1 + 2k + l - 1$$

$$= \frac{(k+l+1)(k+l-2)}{2} + k(l+1) + 1,$$

with which (1) is proved.

We must note that from (1) there follows:

$$P(k,1) = \frac{k(k+1)}{2},$$

$$P(k,k) = 3k^2 - 3k + 1,$$

i.e., we obtain the formulae of the triangular and of the regular hexagonal numbers.

2. Now, we shall generalise the concepts of (k,l)-HRH and (k,l)-HRH number, constructing a hexagon whose sides' lengths satisfy the following conditions: $|AB| = |DE|$, $|BC| = |EF|$ and $|CD| = |FA|$ and all of its angles equal of $120°$.

The number $P(k,l,m)$ of the circles in the (k,l,m)-HRH corresponds to the (k,l,m)-th HRH number. Now, we shall prove the following

THEOREM 2: For every three natural numbers k, l and m:

$$P(k,l,m) = kl + lm + mk - (k + l + m) + 1.$$

The proof of this theorem is analogous to the above one.

If $k = 2, l = 3$ and $m = 4$ the (k,l,m)-HRH has the representation shown on Fig. 4.

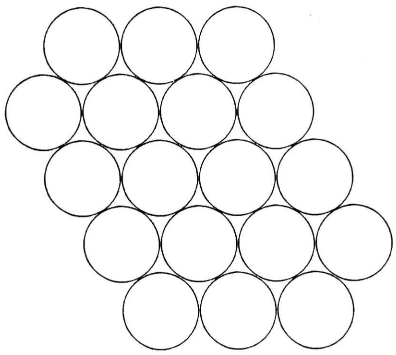

Fig. 4.

Immediatelly we can see that

$$P(k,k,k) = 3k^2 - 3k + 1,$$

i.e., we obtain the formula of the regular hexagon numbers again.

REFERENCE:

[1] Tonkov T., Figure Numbers, Nauka i izkustvo, Sofia, 1971 (in Bulgarian).