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"INCOMPLETE" JACOBSTHAL-TYPE NUMBERS

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1. INTRODUCTION AND PRELIMINARIES

A natural extension of the ideas explored in in [2] would be that of investigating properties of the *incomplete* terms of the generalized sequences defined by the recurrence $W_{n+2} = \mu W_{n+1} + mW_n$, with arbitrary initial conditions. Nevertheless, for the reason explained at the end of this section, we shall confine ourselves to considering the incomplete terms of the Jacobsthal-type sequences $\{G_n^{(m)}\}$ and $\{H_n^{(m)}\}$ ($\mu = 1$ in the above recurrence) whose main properties have been established in [1]. In fact, we shall parallel the arguments of [2] to discover the main properties of the *incomplete Jacobsthal-type numbers* $G_n^{(m)}(k)$ and the *incomplete Jacobsthal-Lucas-type numbers* $H_n^{(m)}(k)$ whose definitions are sketched below, and are formally given in Section 2.

Recall that the numbers $G_n^{(m)}$ and $H_n^{(m)}$ obey the second-order recurrence relation

$$A_n^{(m)} = A_{n-1}^{(m)} + mA_{n-2}^{(m)} \quad (m \text{ a positive integer}) \tag{1.1}$$

(here A stands for either G or H), with initial conditions $G_0^{(m)} = 0$, $G_1^{(m)} = H_1^{(m)} = 1$ and $H_0^{(m)} = 2$. Using standard methods (e.g., see (2.1)-(2.5) of [3]) yields the following closed-form expressions (Binet forms) for these numbers:

$$G_n^{(m)} = \frac{\alpha_m^n - \beta_m^n}{\Delta_m} \quad \text{and} \quad H_n^{(m)} = \alpha_m^n + \beta_m^n \tag{1.2}$$

where

$$\Delta_m = \sqrt{4m+1}\ , \quad \alpha_m = (1+\Delta_m)/2\ , \quad \beta_m = (1-\Delta_m)/2 \tag{1.3}$$

so that

$$\alpha_m + \beta_m = 1 \ , \ \alpha_m - \beta_m = \Delta_m \ , \ \alpha_m \beta_m = -m \, . \tag{1.4}$$

Moreover, we recall that combinatorial expressions for the numbers in question are

$$G_n^{(m)} = \sum_{r=0}^{\tilde{n}} m^r \mathrm{B}_r^{n-1-r} \qquad (n \geq 1), \tag{1.5}$$

$$H_n^{(m)} = \sum_{r=0}^{\hat{n}} \frac{nm^r}{n-r} \mathrm{B}_r^{n-r} \qquad (n \geq 1) \tag{1.6}$$

where

$$\tilde{n} = \lfloor (n-1)/2 \rfloor \quad \text{and} \quad \hat{n} = \lfloor n/2 \rfloor , \tag{1.7}$$

$\mathrm{B}_m^h = \binom{h}{m}$, and the symbol $\lfloor \cdot \rfloor$ denotes the greatest integer function. Induction on n provides the required proofs of (1.5) and (1.6). Another combinatorial expression for $H_n^{(m)}$ is given in (2.6) of [1]. Namely, we have

$$H_n^{(m)} = \frac{1}{2^{n-1}} \sum_{r=0}^{\hat{n}} \Delta_m^{2r} \mathrm{B}_{2r}^n \, . \tag{1.8}$$

From (1.1), we can readily observe that, as special cases, one has

$$G_n^{(1)} = F_n \quad \text{and} \quad H_n^{(1)} = L_n , \tag{1.9}$$

$$G_n^{(2)} = J_n \quad \text{and} \quad H_n^{(2)} = j_n , \tag{1.10}$$

where F_n , L_n , J_n and j_n are the Fibonacci, Lucas, Jacobsthal and Jacobsthal-Lucas (see [4]) numbers, respectively. *Vice versa*, the numbers defined by (1.2) [or by (1.5) and (1.6)], with m arbitrary, can be viewed as a generalization of (1.10), whence the title of this article. A further special case arises when m is a *pronic* number [$m = h(h + 1)$]. In fact, from (1.2) and (1.3), we get the expressions

$$G_n^{(m)} = \frac{(h+1)^n - (-h)^n}{2h+1} \quad \text{and} \quad H_n^{(m)} = (h+1)^n + (-h)^n , \tag{1.11}$$

which, for $h = 1$, give the Binet forms for J_n and j_n (see (2.3) and (2.4) of [4]).

The numbers $G_n^{(m)}(k)$ and $H_n^{(m)}(k)$, which are the object of our study, are obtained by letting the upper range indicators (the parameter k) of the summations on the r.h.s. of (1.5) and (1.6) vary from 0 to $\tilde{n}$ and $\hat{n}$, respectively. It is worth mentioning that these numbers enjoy quite interesting congruence properties the simplest of which are illustrated in Subsection 5.1. A more thorough study of the congruence properties of $H_n^{(m)}(k)$ led to a supposedly new characterization of prime numbers. This discovery will be the object of a future paper of which the present one can be seen as a prologue, and motivates our confining the extension of [2] to the sequences investigated in [1].

2. DEFINITIONS OF THE NUMBERS $G_n^{(m)}(k)$ AND $H_n^{(m)}(k)$

After defining the integers $G_n^{(m)}(k)$ and $H_n^{(m)}(k)$, we show their explicit expressions for certain special values of k. As an illustration, Tables 1 and 2 display them (with $m=5$) for the first few values of n.

Definition 1. Let the *incomplete Jacobsthal-type numbers* $G_n^{(m)}(k)$ be defined as

$$G_n^{(m)}(k) \overset{\text{def}}{=} \sum_{r=0}^{k} m^r \mathrm{B}_r^{n-1-r} \quad (n = 1, 2, 3, \ldots; 0 \le k \le \tilde{n}). \tag{2.1}$$

Definition 2. Let the *incomplete Jacobsthal-Lucas-type numbers* $H_n^{(m)}(k)$ be defined as

$$H_n^{(m)}(k) \overset{\text{def}}{=} \sum_{r=0}^{k} \frac{nm^r}{n-r} \mathrm{B}_r^{n-r} \quad (n = 1, 2, 3, \ldots; 0 \le k \le \hat{n}). \tag{2.2}$$

Some special cases of (2.1) are

$$G_n^{(m)}(0) = 1 \quad (n \ge 1), \tag{2.3}$$

$$G_n^{(m)}(1) = 1 + m(n-2) \quad (n \ge 3), \tag{2.4}$$

$$G_n^{(m)}(2) = 1 + m[m(n^2 - 7n + 12) + 2n - 4] / 2 \quad (n \ge 5), \tag{2.5}$$

$$G_n^{(m)}(\tilde{n}) = G_n^{(m)} \quad (n \ge 1), \tag{2.6}$$

$$G_n^{(m)}(\tilde{n} - 1) = G_n^{(m)} - \begin{cases} nm^{(n-2)/2} / 2 & (n \ge 4, \text{ even}) \\ m^{(n-1)/2} & (n \ge 3, \text{ odd}), \end{cases} \tag{2.7}$$

$$G_n^{(m)}(\tilde{n} - 2) = G_n^{(m)} - \begin{cases} nm^{(n-4)/2}(24m + n^2 - 4) / 48 & (n \ge 6, \text{ even}) \\ m^{(n-3)/2}(8m + n^2 - 1) / 8 & (n \ge 5, \text{ odd}), \end{cases} \tag{2.8}$$

whereas some special cases of (2.2) are

$$H_n^{(m)}(0) = 1 \quad (n \ge 1), \tag{2.9}$$

$$H_n^{(m)}(1) = 1 + nm \quad (n \geq 2), \tag{2.10}$$

$$H_n^{(m)}(2) = 1 + nm[m(n-3)+2]/2 \quad (n \geq 4), \tag{2.11}$$

$$H_n^{(m)}(\hat{n}) = H_n^{(m)} \quad (n \geq 1), \tag{2.12}$$

$$H_n^{(m)}(\hat{n}-1) = H_n^{(m)} - \begin{cases} 2m^{n/2} & (n \geq 2, \text{ even}) \\ nm^{(n-1)/2} & (n \geq 3, \text{ odd}), \end{cases} \tag{2.13}$$

$$H_n^{(m)}(\hat{n}-2) = H_n^{(m)} - \begin{cases} m^{(n-2)/2}(8m+n^2)/4 & (n \geq 4, \text{ even}) \\ nm^{(n-3)/2}(24m+n^2-1)/24 & (n \geq 5, \text{ odd}). \end{cases} \tag{2.14}$$

The numbers $G_n^{(5)}(k)$ and $H_n^{(5)}(k)$ are shown below for the first few values of n and the corresponding admissible values of k.

Table 1. The numbers $G_n^{(5)}(k)$ for $1 \leq n \leq 10$

$n \backslash k$	0	1	2	3	4
1	1				
2	1				
3	1	6			
4	1	11			
5	1	16	41		
6	1	21	96		
7	1	26	176	301	
8	1	31	281	781	
9	1	36	411	1661	2286
10	1	41	566	3066	6191

Table 2. The numbers $H_n^{(5)}(k)$ for $1 \leq n \leq 10$

$n \backslash k$	0	1	2	3	4	5
1	1					
2	1	11				
3	1	16				
4	1	21	71			
5	1	26	151			
6	1	31	256	506		
7	1	36	386	1261		
8	1	41	541	2541	3791	
9	1	46	721	4471	10096	
10	1	51	926	7176	22801	29051

3. PROPERTIES OF THE NUMBERS $G_n^{(m)}(k)$ AND $H_n^{(m)}(k)$

Most of the properties established in this section pertain to sums of the numbers under study along the diagonals, rows, and columns of the triangular arrays displayed in Tables 1 and 2. Their proofs will be given in Section 4.

Warning. To save space, whenever $G_n^{(m)}(k)$ and $H_n^{(m)}(k)$ enjoy the same property, it will be stated only once, and the symbol A will stand for either G or H [cf. (1.1)]. In general, the validity of these properties is subject to certain conditions on the value of k: they can readily be derived from the conditions on k in (2.1) and (2.2). For example, Proposition 2 holds for $0 \le k \le (n-h-1)/2$ (if $A := G$), and for $0 \le k \le (n-h)/2$ (if $A := H$).

Proposition 1. $A_{n+2}^{(m)}(k+1) = A_{n+1}^{(m)}(k+1) + mA_n^{(m)}(k)$ (recurrence relation). (3.1)

By using (2.1), (2.2) and the basic recurrence relation for binomial coefficients [6, p. 1], (3.1) can be readily transformed into the non-homogeneous second-order recurrence relations

$$A_{n+2}^{(m)}(k) = A_{n+1}^{(m)}(k) + mA_n^{(m)}(k) - \begin{cases} m^{k+1}\mathrm{B}_k^{n-1-k} & \text{if } A := G \\ m^{k+1}n\mathrm{B}_k^{n-k}/(n-k) & \text{if } A := H. \end{cases} \tag{3.2}$$

The relation (3.1) generalizes beautifully as follows.

Proposition 2. $$\sum_{r=0}^{h} m^{h-r}\mathrm{B}_r^h A_{n+r}^{(m)}(k+r) = A_{n+2h}^{(m)}(k+h). \tag{3.3}$$

Remark. Relations (3.1) and (3.3) coincide for $h = 1$.

Proposition 3. $$\sum_{r=0}^{h} A_{n+r}^{(m)}(k) = \frac{A_{n+h+1}^{(m)}(k+1) - A_{n+1}^{(m)}(k+1)}{m}. \tag{3.4}$$

The main relation between $G_n^{(m)}(k)$ and $H_n^{(m)}(k)$ is then stated.

Proposition 4. $$H_n^{(m)}(k) = mG_{n-1}^{(m)}(k-1) + G_{n+1}^{(m)}(k). \tag{3.5}$$

Note that, from (3.5) and (3.1) (for $A := G$), one gets immediately the equivalent relation

$$H_n^{(m)}(k) = 2mG_{n-1}^{(m)}(k-1) + G_n^{(m)}(k) \tag{3.5'}$$

whence

$$\begin{cases} H_n^{(m)}(k) - G_n^{(m)}(k) = 2mG_{n-1}^{(m)}(k-1) \\ H_n^{(m)}(k) + G_n^{(m)}(k) = 2G_{n+1}^{(m)}(k) . \end{cases} \tag{3.6}$$

Finally, let $R_n^{(m)}$ (resp. $S_n^{(m)}$) denote the sum of all entries of the n-th row of the array in Table 1 (resp. Table 2).

Proposition 5. $$R_n^{(m)} \stackrel{\text{def}}{=} \sum_{k=0}^{\tilde{n}} G_n^{(m)}(k) = \frac{[m(4\tilde{n}+6)+\tilde{n}+1]G_n^{(m)} - mnH_{n-1}^{(m)}}{4m+1} . \tag{3.7}$$

Proposition 6. $$S_n^{(m)} \stackrel{\text{def}}{=} \sum_{k=0}^{\hat{n}} H_n^{(m)}(k) = (\hat{n}+1)H_n^{(m)} - mnG_{n-1}^{(m)} . \tag{3.8}$$

4. PROOFS

Proof of Proposition 1 (for A := G). Use (2.1) to rewrite the r.h.s. of (3.1) as

$$\sum_{r=0}^{k+1} m^r \mathrm{B}_r^{n-r} + \sum_{r=0}^{k} m^{r+1} \mathrm{B}_r^{n-1-r} = \sum_{r=0}^{k+1} m^r \mathrm{B}_r^{n-r} + \sum_{r=1}^{k+1} m^r \mathrm{B}_{r-1}^{n-r}$$

$$= -m^0 \mathrm{B}_{-1}^{n} + \sum_{r=0}^{k+1} m^r \left[\mathrm{B}_r^{n-r} + \mathrm{B}_{r-1}^{n-r}\right] = -0 + \sum_{r=0}^{k+1} m^r \mathrm{B}_r^{n+1-r} = G_{n+2}^{(m)}(k+1) \quad \blacksquare$$

Proof of Proposition 2 (for A := G). Let us use induction on h . Identity (3.3) holds clearly for $h = 0$ (trivially) and $h = 1$ [see (3.1)]. Suppose it holds for a certain h > 1. For the inductive step $h := h + 1$, write

$$\sum_{r=0}^{h+1} m^{h+1-r} \mathrm{B}_r^{h+1} G_{n+r}^{(m)}(k+r) = \sum_{r=0}^{h+1} m^{h+1-r} \left[\mathrm{B}_r^{h} + \mathrm{B}_{r-1}^{h}\right] G_{n+r}^{(m)}(k+r)$$

$$= m^0 \mathrm{B}_{h+1}^{h} G_{n+h+1}^{(m)}(k+h+1) + m \sum_{r=0}^{h} m^{h-r} \mathrm{B}_r^{h} G_{n+r}^{(m)}(k+r)$$

$$+ \sum_{r=-1}^{h} m^{h-r} \mathrm{B}_r^{h} G_{n+r+1}^{(m)}(k+r+1)$$

$$= 0 + mG_{n+2h}^{(m)}(k+h) \qquad \text{[by the inductive hypothesis (i.h., for short)]}$$

$$+ m^{h+1} \mathrm{B}_{-1}^{h} G_n^{(m)}(k) + \sum_{r=0}^{h} m^{h-r} \mathrm{B}_r^{h} G_{n+r+1}^{(m)}(k+r+1)$$

$$= 0 + mG^{(m)}_{n+2h}(k+h) + 0 + G^{(m)}_{n+1+2h}(k+h+1) \qquad \text{(by the i.h.)}$$

$$= G^{(m)}_{n+2+2h}(k+h+1) = G^{(m)}_{n+2(h+1)}(k+h+1) \qquad \text{[from (3.1)]} \ \blacksquare$$

The proofs of Propositions 1 and 2 for $A := H$ can be obtained in a similar way by using the combinatorial identity (see [6, p. 64])

$$\frac{n}{n-r} \mathrm{B}^{n-r}_{r} = \mathrm{B}^{n-r}_{r} + \mathrm{B}^{n-1-r}_{r-1} . \tag{4.1}$$

Proof of Proposition 3 (Hint). Use induction on h and (3.1) ■

Proof of Proposition 4. Use (2.1) to rewrite the r.h.s. of (3.5) as

$$\sum_{r=0}^{k-1} m^{r+1} \mathrm{B}^{n-2-r}_{r} + \sum_{r=0}^{k} m^{r} \mathrm{B}^{n-r}_{r} = \sum_{r=1}^{k} m^{r} \mathrm{B}^{n-1-r}_{r-1} + \sum_{r=0}^{k} m^{r} \mathrm{B}^{n-r}_{r}$$

$$= -m^{0} \mathrm{B}^{n-1}_{-1} + \sum_{r=0}^{k} m^{r} \left[\mathrm{B}^{n-1-r}_{r-1} + \mathrm{B}^{n-r}_{r}\right]$$

$$= -0 + \sum_{r=0}^{k} \frac{nm^{r}}{n-r} \mathrm{B}^{n-r}_{r} = H^{(m)}_{n}(k) \qquad \text{[from (4.1) and (2.2)]} \ \blacksquare$$

To prove Propositions 5 and 6, we need the identities

$$\sum_{r=0}^{\tilde{n}} m^{r} r \mathrm{B}^{n-1-r}_{r} = \frac{m[nH^{(m)}_{n-1} - 2G^{(m)}_{n}]}{4m+1} \tag{4.2}$$

and

$$\sum_{r=0}^{\hat{n}} r \frac{nm^{r}}{n-r} \mathrm{B}^{n-r}_{r} = mnG^{(m)}_{n-1} . \tag{4.3}$$

Proof of (4.2). By means of the same technique as that used in [5], first replace m by the indeterminate x in (1.1)-(1.5), then write

$$\frac{d}{dx} G^{(x)}_{n} = [n(\alpha^{n-1}_{x} + \beta^{n-1}_{x}) - \frac{2(\alpha^{n}_{x} - \beta^{n}_{x})}{\Delta_{x}}] / \Delta^{2}_{x} \qquad \text{[from (1.2) and (1.3)]}$$

$$= [nH^{(x)}_{n-1} - 2G^{(x)}_{n}] / (4x+1) . \tag{4.4}$$

and

$$\frac{d}{dx} G_n^{(x)} = \sum_{r=0}^{\tilde{n}} x^{r-1} r \mathrm{B}_r^{n-1-r} \qquad \text{[from (1.5)].} \tag{4.5}$$

Equating the r.h.s. of (4.4) and (4.5) and letting $x = m$ therein, yields (4.2) ∎
The proof of (4.3) is similar and is omitted for brevity.

Proof of Proposition 5. From (2.1) and the l.h.s. of (3.7), write

$$R_n^{(m)} = G_n^{(m)}(0) + G_n^{(m)}(1) + \cdots + G_n^{(m)}(\tilde{n})$$

$$= \left\{m^0 \mathrm{B}_0^{n-1}\right\} + \left\{m^0 \mathrm{B}_0^{n-1} + m^1 \mathrm{B}_1^{n-2}\right\} + \cdots$$

$$\cdots + \left\{m^0 \mathrm{B}_0^{n-1} + m^1 \mathrm{B}_1^{n-2} + m^2 \mathrm{B}_2^{n-3} + \cdots + m^{\tilde{n}} \mathrm{B}_{\tilde{n}}^{n-1-\tilde{n}}\right\}$$

$$= \sum_{r=0}^{\tilde{n}} (\tilde{n} + 1 - r) m^r \mathrm{B}_r^{n-1-r} = (\tilde{n} + 1) G_n^{(m)} - \sum_{r=0}^{\tilde{n}} m^r r \mathrm{B}_r^{n-1-r} \qquad \text{[from (1.5)]}$$

$$= (\tilde{n} + 1) G_n^{(m)} - \frac{m[n H_{n-1}^{(m)} - 2 G_n^{(m)}]}{4m + 1} \qquad \text{[from (4.2)],}$$

whence the desired result ∎

Proof of Proposition 6. By using (2.2) to rearrange the addends of the sum on the l.h.s. of (3.8) (cf. the proof of Proposition 5) it is not hard to see that

$$S_n^{(m)} = \sum_{r=0}^{\hat{n}} (\hat{n} + 1 - r) \frac{nm^r}{n - r} \mathrm{B}_r^{n-r} = (\hat{n} + 1) H_n^{(m)} - \sum_{r=0}^{\hat{n}} r \frac{nm^r}{n - r} \mathrm{B}_r^{n-r} \qquad \text{[from (1.6)]}$$

whence one gets (3.8) by virtue of (4.3) ∎

5. CONCLUDING COMMENTS AND FURTHER RESULTS

The properties of the numbers $G_n^{(m)}(k)$ and $H_n^{(m)}(k)$ are by no means exhausted by the brief account given in this article. As a minor instance, we urge the interested reader to prove the identity

$$\sum_{k=0}^{h} \left[A_{n+2k}^{(m)}(k) + (m - 1) A_{n+2k+1}^{(m)}(k)\right] = m A_{n+2h+1}^{(m)}(k) \,. \tag{5.1}$$

5.1. Some simple congruence properties of $A_n^{(m)}(k)$

The numbers $A_n^{(m)}(k)$ are clearly congruent to 1 modulo m by virtue of their definitions (2.1) and (2.2). This implies that, for m even, they are odd.

Proposition 7. If m is an arbitrary natural number and $n = 2^h$ $(h = 0, 1, 2, ...)$, then $A_n^{(m)}(k)$ is odd for all admissible values of k.

Proof [for A := G]. The statement is true for $h = 0$ and 1 since, for $n = 2^0$ and 2^1, we have

$$\tilde{n} = 0, 0 \le k \le 0, \text{ and } G_1^{(m)}(0) = G_2^{(m)}(0) = 1 \quad \text{[cf. (2.3)]}.$$

For $h \ge 2$, replace n by 2^h in (2.1) and write

$$G_{2^h}^{(m)}(k) = 1 + \sum_{r=1}^{k} m^r \mathrm{B}_r^{2^h-1-r} \qquad (1 \le k \le 2^{h-1} - 1)$$

whence it is sufficient to prove that

$$\mathrm{B}_r^{2^h-1-r} \equiv 0 \pmod 2 \quad \text{for } 1 \le r \le 2^{h-1} - 1 \quad (h \ge 2). \tag{5.2}$$

The proof of congruence (5.2) is based on a theorem of Singmaster [7], and is available in the proof of Proposition 5 of [2] ■

The proof for $A := H$ is similar (see the proof of Proposition 12 of [2]) and is omitted.

Proposition 8. For all admissible values of k, $A_n^{(m)}(k) \equiv 1 \pmod{m^2}$ if $n \equiv h$ (mod m), where $h = 0$ (resp. 2) for $A := H$ (resp. G).

Proof . From (2.2) and (2.1), write

$$H_n^{(m)}(k) = 1 + mn + \sum_{r=2}^{k} \frac{nm^r}{n-r} \mathrm{B}_r^{n-r} \equiv 1 \pmod{m^2} \text{ if } n \equiv 0 \pmod m$$

and

$$G_n^{(m)}(k) = 1 + m(n-2) + \sum_{r=2}^{k} m^r \mathrm{B}_r^{n-1-r} \equiv 1 \pmod{m^2} \text{ if } n \equiv 2 \pmod m \quad ■$$

Finally, we leave the proofs of the following congruences as an exercise for the interested reader:

$$A_n^{(hn)}(k) \equiv 1 \pmod n \ \forall k \quad (h = 1, 2, 3, \ldots), \tag{5.3}$$

$$A_n^{(m)}(k-1)\, A_{n+1}^{(m)}(k) - A_n^{(m)}(k)\, A_{n+1}^{(m)}(k-1) \equiv 0 \pmod{m^k}. \tag{5.4}$$

5.2. A by-product result

Letting $m = 2$ in (1.3) and (1.8), and $h = 1$ (i.e., $m = 2$) in (1.11) allows us to state the following proposition.

Proposition 9. If p is an odd prime and M_p is a Mersenne prime, then

$$\sum_{r=0}^{(p-1)/2} 9^r \, B_{2r}^p \quad \text{is a perfect number.} \tag{5.5}$$

Formula (5.5) produces all even perfect numbers greater than 6, and shows clearly that all such numbers are congruent to 1 modulo 9. This combinatorial expression for perfect numbers is supposedly new and might be of some interest: at least, this is the opinion of the author of [8].

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