NNTDM 4 (1998) 2, 47-56

ANTHONY SHANNON'S RESEARCH ON FIBONACCI-LIKE OBJECTS

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Devoted to Prof. Anthony Shannon
for his 60-th birthday

Professor Anthony Shannon, or as all his friends call him - Tony Shannon, has carried out a lot of research in different areas of mathematics: number theory, mathematical modeling, fuzzy sets and others, but here we shall discuss only those, which are related to the area of the Fibonacci-like objects. They appear in more than 40 papers published in "The Fibonacci Quarterly" in about a 30-year period, as well as in other journals which we shall not discuss here.

In general, Shannon's papers can be classified into the following 4 groups:

• recurrent sequences and recurrent relations: [4, 5, 7, 9, 11, 14, 18, 19, 20, 21, 23, 24, 25, 26, 28, 29, 30, 33, 44, 49, 51, 61]

• extensions of Fibonacci and Fibonacci-like numbers: [1, 6, 8, 17, 20, 21, 22, 28, 32, 34, 38, 43, 45, 40, 47, 54, 63]

• congruences on Fibonacci numbers: [12, 52, 57]

• other problems related to Fibonacci numbers: reciprocals of Fibonacci numbers [1, 58], generalizations of the Pythagorean theorem [2, 48], Farey-Fibonacci fractions [16], generating functions [3, 10, 35], research on Fermat's last theorem [27], Fibonacci and Lucas curves [37], Fibonacci numbers and Diophantine quadruples [42], generalized Fibonacci continued fractions [38], Asveld's polynomials [39], convolution trees [41, 46, 59, 64], some infinite series [53], some summation identities [55], a generalization of the Catalan's identity [56], relation between fuzzy sets and Fibonacci sequence [64], Fibonacci analogs of the classical polynomials [15], functional equations [62], the Jacobi-Perron algorithm [7, 13], generalization of identities of Catalan and others [36], divisibility properties of Fibonacci numbers [31], Fibonacci model of infectious disease [60].

Below we shall formulate some of the more interesting results from the above papers which do not require the introduction of a large amount of additional information. Some of these papers are reviewed by the author in "Zentralblatt fuer Mathematik" (Germany). The opinion of the author is that the above cited papers illustrate Tony Shannon's aspiration for elegant and graceful mathematical miniatures.

In [6] there is a discussion of the sequence $\{w_n\}$, which ¿satisfies the conditions

$$w_0 = a, w_1 = b,$$

$$w_n = pw_{n-1} - qw_{n-2} \ (n \geq 2),$$

where p, q arbitrary integers. It is proved that

$$(\frac{p}{q^2} w_n w_{n+3})^2 + (2Pw_{n+2}(Pw_{n+2} - w_n))^2 = (w_n^2 + 2Pw_{n+2}(Pw_{n+2} - w_n))^2,$$

where $P = \frac{p^2-q}{2q^2}$.

A generalized third order recurrence relation

$$r_n = Pr_{n-1} + Qr_{n-2} + Rr_{n-3} \ (n \geq 3),$$

where P, Q, R are arbitrary integers, is discussed in [5]. Let the sequence $\{p_n\} \equiv \{r_n\}$ and $p_0 = a, p_1 = b, p_2 = c$ for integers a, b, c; the sequence $\{s_n\} \equiv \{r_n\}$ and $s_0 = 1, s_1 = P, s_2 = P^2 + Q$, and the sequence $\{t_n\} \equiv \{r_n\}$ and $t_0 = 3, t_1 = P, t_2 = P^2 + Q$. Then

$$p_n = bs_{n-1} + (c - bP)s_{n-2} + aRr_{n-3} \ (n \geq 3),$$

$$s_n = \sum_{m=0}^{[n/2]} \sum_{r=0}^{[n/3]} \binom{n-m-2r}{m+r} \binom{m+r}{r} P^{n-2m-3r} Q^m R^r$$

$$t_n = \sum_{m=0}^{[n/2]} \sum_{r=0}^{[n/3]} \frac{n}{n-m-2r} \binom{n-m-2r}{m+r} \binom{m+r}{r} P^{n-2m-3r} Q^m R^r \ (n \geq 0).$$

In [8] there is a definition of the sequence

$$W^r_{s,n+r} = \sum_{j=1}^{r} (-1)^{j+1} P_{rj} W^r_{s,n+r-j} \ (s = 0, 1, ..., r-1; n \geq r \geq 1),$$

where P_{rj} are arbitrary integers and $W^r_{s,n}$ $(n = 0, 1, ..., r-1)$ have suitable initial values. Let $w = exp(\frac{2i\pi}{r}), i^2 = -1$, and

$$d = \begin{bmatrix} 1 & 1 & \dots & 1 \\ a_{r,1} & a_{r,2} & \dots & a_{r,r} \\ \vdots & \vdots & & \vdots \\ a^{r-1}_{r,1} & a^{r-1}_{r,2} & \dots & a^{r-1}_{r,r} \end{bmatrix},$$

and let $S^n_m = \sum_{\Sigma k = m} \prod_{i=1}^{n} W^r_{k,x_i+r}$ for $k = 0, 1, ..., r-1$, where the sum is on all permutations of $\{x_1, x_2, ..., x_n\}$. Then

$$W^r_{s,x_1+x_2+...+x_n+r} = r^{-n} \sum_{j=0}^{r-1} \sum_{k=0}^{(r-1)n} (dw^{-j})^{k-s} S^n_k.$$

If

$$u_n = \begin{cases} \sum_{j=1}^{r} (-1)^{j+1} P_j u_{n-j}, & \text{if } (n > 0) \\ 1, & \text{if } (n = 0) \\ 0, & \text{if } (n < 0) \end{cases}$$

and

$$v_n = \begin{cases} \sum_{j=1}^{r} (-1)^{j+1} P_j v_{n-j}, & \text{if } (n > r) \\ \sum_{\alpha^n_j}^{r}, & \text{if } (0 \leq n \leq r) \\ 0, & \text{if } (n < 0) \end{cases}$$

where the P_j are arbitrary integers and the α_j are the roots of the equation

$$x^r - \sum_{j=1}^{r} (-1)^{j+1} P_j u_{r-j} = 0,$$

then $\{u_n\}$ and $\{v_n\}$ are called in [15] generalized Fibonacci and Lucas numbers, respectively. There it is proved that

$$\sum_{n=0}^{\infty} u_n x^n = exp(\sum_{m=1}^{\infty} \frac{v_m x^m}{m}).$$

Let

$$u(x) = exp(\sum_{m=1}^{\infty} \frac{v_m x^m}{m}).$$

Then

$$\sum_{n=0}^{\infty} \frac{u_n(x) t^n}{n!} = exp(xt + \sum_{m=1}^{\infty} \frac{v_m (x)^m}{m}),$$

from where it follows that

$$u_n(0) = u_n n!$$

and

$$\sum_{n=0}^{\infty} \frac{u_n(x) t^n}{n!} = e^{xt} \sum_{n=1}^{\infty} \frac{u_n(0) t^n}{n!}.$$

Let $U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $V_n = \alpha^n + \beta^n$, where $\alpha = \frac{p+\sqrt{p^2-4q}}{2}$, and $\beta = \frac{p-\sqrt{p^2-4q}}{2}$ are the roots, assumed distinct, of

$$x^2 - px + q = 0.$$

Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

In [31] a sequence is introudced which is defined by the recurrence relation

$$F_{N,n} = F_{N,n-1} + N F_{N,n-2},$$

where $N > 0$ and $n > 2$ are integers, $F_{N,1} = F_{N,2} = 1$, which has also the form

$$F_{N,n} = (a^n - b^n)/(a - b),$$

where a, b are zeros of the equation $x^2 - x - N = 0$, and for it the following equality is proved:

$$\frac{F_{N,k(n+1)}}{F_{N,k}} = \sum_{0 \le r+s \le n} \binom{r}{s} \binom{n-r}{s} F_{N,k-1}^{r-s} F_{N,k}^{2s} F_{N,k+1}^{n-r-s} N^r.$$

In [36] it is proved that for the generalized Fibonacci-type second order recurrence sequences $\{w_n\}$ and $\{u_n\}$, for which $w_0 = a, w_1 = b$, a, b, p, q are integers, the integers

$$4 w_n w_{n+r} w_{n+s} w_{n+r+s} + e^2 q^{2n} u_r^2 u_s^2$$

and

$$4w_n^2 w_{n+r}w_{n+s}w_{n+r+s+t} + \frac{1}{9}e^2 q^{2(n+1)}(C + w_n D)^2$$

are perfect squares, where

$$C = u_{n+r}u_s u_t + u_{n+s}u_t u_r + u_{n+t}u_r u_s$$

and

$$D = u_r u_{s+t} + u_s u_{t+r} + u_t u_{r+s},$$

$$e = pab - qa^2 - b^2.$$

The following assertions are proved in [49]:

• The equality

$$nx^2 + (n + r)x - (n + 2r) = 0$$

has rational solutions if and only if $n = r(F_{2m+1} - 1)$ where m is an integer. The solutions $\frac{F_{2m}}{F_{2m+1}-1}$ and $-\frac{F_{2m+2}}{F_{2m+1}-1}$ are independent of r.

• If $5|(L_{2m+1} - 4)$, the equality

$$nx^2 + (n + 2r)x - (n + r) = 0$$

has rational solutions if and only if $n = \frac{r(L_{4m+3}-4)}{5} = rF_{2m}F_{2m+3}$ where m is an integer. The solutions are $\frac{F_{2m+2}}{F_{2m+3}} = \frac{L_{4m+2}-3}{L_{4m+3}-4}$ and $-\frac{F_{2m+1}}{F_{2m}} = -\frac{L_{4m+4}+3}{L_{4m+3}-4}$.

• If $5\not|(L_{2m+1} - 4)$, the above equality has rational solutions if and only if $r = 5s$, s is an integer, and $n = s(L_{4m+1} - 4)$. The solutions are $\frac{L_{4m}-3}{L_{2m+1}-4}$ and $-\frac{L_{2m+2}+3}{L_{2m+1}-4}$.

• If $5|(L_{2m+1} - 1)$, the equality

$$nx^2 + (n - r)x - (n + r) = 0$$

has rational solutions if and only if $n = \frac{r(L_{4m+1}-1)}{5} = rF_{2m}F_{2m+1}$ where m is an integer. The solutions are $\frac{F_{2m-1}}{F_{2m}} = \frac{L_{4m}+3}{L_{4m+1}-1}$ and $-\frac{F_{2m+2}}{F_{2m+1}} = -\frac{L_{4m+2}-3}{L_{4m+1}-1}$.

• If $5\not|(L_{2m+1} - 1)$, the above equality has rational solutions if and only if $r = 5s$, s is an integer, and $n = s(L_{2m+1} - 1)$. The solutions are $\frac{L_{2m}+3}{L_{2m+1}-1}$ and $-\frac{L_{2m+2}-3}{L_{2m+1}-1}$.

• If $5|(L_{2m+1} + 1)$, the equality

$$nx^2 + (n + r)x - (n - r) = 0$$

has rational solutions if and only if $n = \frac{r(L_{4m+3}+1)}{5} = rF_{2m+1}F_{2m+2}$ where m is an integer. The solutions are $\frac{F_{2m}}{F_{2m+1}} = \frac{L_{4m+2}-3}{L_{4m+3}+1}$ and $-\frac{F_{2m+3}}{F_{2m+2}} = -\frac{L_{4m+4}+3}{L_{4m+3}+1}$.

• If $5\not|(L_{2m+1} + 1)$, the above equality has rational solutions if and only if $r = 5s$, s is an integer, and $n = s(L_{2m+1} + 1)$. The solutions are $\frac{L_{2m}-3}{L_{2m+1}+1}$ and $-\frac{L_{2m+2}+3}{L_{2m+1}+1}$.

• If $5|(L_{2m+1} + 4)$, the equality

$$nx^2 + (n - 2r)x - (n - r) = 0$$

has rational solutions if and only if $n = \frac{r(L_{4m+1}+4)}{5} = rF_{2m-1}F_{2m+2}$ where m is an integer. The solutions are $\frac{F_{2m+1}}{F_{2m+2}} = \frac{L_{4m}+3}{L_{4m+1}+4}$ and $-\frac{F_{2m}}{F_{2m-1}} = -\frac{L_{2m+2}-3}{L_{2m+1}+4}$.

• If $5\not|(L_{2m+1} - 1)$, the above equality has rational solutions if and only if $r = 5s$, s is an integer, and $n = s(L_{2m+1} + 4)$. The solutions are $\frac{L_{2m}+3}{L_{2m+1}+4}$ and $-\frac{L_{2m+2}-3}{L_{2m+1}+4}$.

• The equality

$$nx^2 + (n - r)x - (n - 2r) = 0$$

has rational solutions if and only if $n = r(F_{2m+1}+1)$ where m is an integer. The solutions are $\frac{F_{2m}}{F_{2m+1}+1}$ and $-\frac{F_{2m+2}}{F_{2m+1}+1}$.

In [53] it is proved that if f has a domain of convergence which includes $x\alpha^k$ and $x\beta^k$, then

$$\sum_{n=0}^{\infty} a_n x^n U_{kn+1} = \frac{\alpha f(x\alpha^k) - \beta f(x\beta^k)}{\sqrt{p^2-4q}},$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn} = \frac{f(x\alpha^k) - f(x\beta^k)}{\sqrt{p^2-4q}},$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn-1} = \frac{\beta f(x\alpha^k) - \alpha f(x\beta^k)}{q\sqrt{p^2-4q}},$$

$$\sum_{n=0}^{\infty} a_n x^n V_{kn} = f(x\alpha^k) + f(x\beta^k).$$

Using the above equalities, the authors obtain a series of interesting trigonometrical equalities. For example,

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1} \mathrm{sin} k(2n+1)\theta}{(2n+1)!} = \cos(x \mathrm{cos} k\theta)\sinh(x \mathrm{sin} k\theta),$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n} \mathrm{sin} 2kn\theta}{(2n)!} = \sin(x \mathrm{cos} k\theta)\sinh(x \mathrm{sin} k\theta),$$

$$\sum_{n=0}^{\infty} \frac{x^{2n+1} \mathrm{sin} k(2n+1)\theta}{(2n)!} = \sin(x \mathrm{sin} k\theta)\cosh(x \mathrm{cos} k\theta),$$

$$\sum_{n=0}^{\infty} \frac{x^{2n} \mathrm{sin} 2kn\theta}{(2n)!} = \sin(x \mathrm{sin} k\theta)\sinh(x \mathrm{cos} k\theta).$$

The following interesting trigonometrical equalities are proved in [54] for the natural numbers k and n, where the above $q = 1$:

$$\tan^{-1} U_{n+2} - \tan^{-1} U_n = \tan^{-1}(\frac{p}{U_{n+1}}), \text{ if } n \text{ is even},$$

$$\tan^{-1}(\frac{1}{U_n}) + \tan^{-1}(\frac{1}{U_{n+2}}) = \tan^{-1}(\frac{V_{n+1}}{U_{n+1}^2}), \text{ if } n \neq -1 mboxisodd,$$

$$\tan^{-1}(\frac{U_n}{U_{n+k}}) - \tan^{-1}(\frac{U_{n-k}}{U_n}) = \begin{cases} \tan^{-1}(\frac{(-1)^n U_k^2}{V_k U_n^2}), & k mboxeven, \\ \tan^{-1}(\frac{(-1)^{n-1} U_k}{U_{2n}}), & k \text{ odd} \end{cases}$$

$$\tan^{-1}(\frac{V_{n-k}}{V_n}) - \tan^{-1}(\frac{V_n}{V_{n+k}}) = \begin{cases} \tan^{-1}(\frac{(-1)^n (p^2+4) U_k^2}{V_k V_n^2}), & k \text{ even}, \\ \tan^{-1}(\frac{(-1)^{n-1} U_k}{U_{2n}}), & k \text{ odd} \end{cases}$$

Similar equalities are proved for the trigonometric function "tanh".

Let the sequences $\{U_n\}$ and $\{V_n\}$ be defined as above Let p and q be fixed real numbers. The matrices $M_{k,m}, X_k$ and $N_{k,m}$ are defined as:

$$M_{k,m} = \begin{bmatrix} U_{k+m} & -q^m U_k \\ U_k & -q^m U_{k-m} \end{bmatrix},$$

$$X_k = \begin{bmatrix} V_k & U_k \\ (p^2 - 4q)U_k & V_k \end{bmatrix},$$

$$N_{k,m} = \begin{bmatrix} V_{k+m} & -q^m V_k \\ V_k & -q^m V_{k-m} \end{bmatrix},$$

Some properties of these three matrices are studied in [55].

The ideas for graph-theory interpretations of the Fibonacci numbers (see [50, 41, 46, 59] will be not discussed here. They are the object of detailed description in a forthcoming book.

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