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ABOUT GOLDBACH'S PROBLEM (III)

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Abstract:

Without any condition, is proved conjecture [A] of Hardy-Littlewood, in their paper P. N. III.

1.- In a former paper (1), the author has proved the following theorem:

Theorem [A].
Let $\nu(t)$ defined by:

$$\nu(t) = \sum_{p_1 + p_2 = t} \log p_1 \log p_2 \qquad [1]$$

Then we have the exact relation:

$$\nu(t) = \mathcal{L}^{-1} \{ e^s (1-e^{-s})^2 \mathcal{L}^2 (\vartheta(u)) \} \qquad [2]$$

where:

$$\vartheta(u) = \sum_{p \le u} \log p$$

p are the prime numbers and $\mathcal{L}(f(u))$ is the usual Laplace transform.

2.- A standard theorem in the theory of the Laplace transform.
The g ralized theorem of the final value runs as follows:

Theorem [B].
If $F(t) \sim G(t)$ when $t \to \infty$, then $f(s) \sim g(s)$ when $s \to o$ where $f(s) = \mathcal{L}\{F(t)\}$ and $g(s) = \mathcal{L}\{G(t)\}$ (and reciprocally).

This holds equally well for continuous and discontinuous functions.

That $F(t) \sim G(t)$ means that:

$$\lim_{t \to \infty} \frac{F(t)}{G(t)} = 1$$

3.- Some examples of the above theorem.

1) It is known that if

$$g(s) = \frac{1}{s(e^s - 1)}$$

then

$$[3] \qquad F(t) = n \qquad n \leq t < n + 1 \qquad n = 0, 1, 2, \ldots$$

But

$$\lim_{s \to o} \frac{1}{s(e^s - 1)} = \frac{1}{s^2} = 1$$

and

$$[4] \qquad \mathcal{L}^{-1} f(s) = \mathcal{L}^{-1} \{ 1 / s^2 \} = t = G(t)$$

It is obvious from [3] and [4] that $F(t) \sim G(t)$.

2) If we have, in change,

$$g(s) = \frac{1}{s(e^s - r)}$$

then

[5] $$F(t) = \sum_{k=1}^{[t]} r^k$$

But

$$\lim_{s \to o} \frac{1}{s(e^s - r)} = \frac{1}{s(1-r)} = f(s)$$

and

[6] $$\mathcal{L}^{-1} f(s) = \mathcal{L}^{-1} \left\{ \frac{1}{s(1-r)} \right\} = \frac{1}{1-r} = G(t)$$

It is evident from [5] and [6] that $F(t) \sim G(t)$. $(|r| < 1)$

3) In the case that

$$g(s) = \frac{\sinh(sx)}{s^2 \sinh(sa)}$$

we have

[7] $$F(t) = \frac{xt}{a} + \frac{2a}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sin \frac{n\pi x}{a} \sin \frac{n\pi t}{a}$$

But

$$\lim_{s \to o} \frac{\sinh(sx)}{s^2 (\sinh(sa))} = \frac{x}{a s^2} = f(s)$$

and

[8] $$\mathcal{L}^{-1} f(s) = \mathcal{L}^{-1} \left\{ \frac{x}{a s^2} \right\} = \frac{x}{a} t = G(t)$$

Again we have $F(t) \sim G(t)$.

The interested reader can also check the theorem, for instance, using the formulas [87], [91], [92], [114], [117] to [123] that appear in Appendix B of the textbook "The Laplace Transform", by M. Spiegel from where are also taken the preceding examples.

4.- We apply now theorem [B] to formula [2].
On the one hand, let $\nu^*(t)$ be a function such that $\nu^*(t) \sim \nu(t)$.
On the other hand,

$$\lim_{s \to o} e^s (1-e^{-s}) = \lim_{s \to o} s^2$$

so that:

$$f(s) = e^s (1-e^{-s})^2 \mathcal{L}^2 (\vartheta(u))$$

and:

$$g(s) = s^2 \mathcal{L}^2 (\vartheta(u))$$

are such that:

$$f(s) \sim g(s)$$

as $s \to o$.
The theorem, when applied to the formula:

$$\mathcal{L} \{ \nu(t) \} = e^s (1-e^{-s})^2 \mathcal{L}^2 \{ \vartheta(u) \}$$

yields:

$$v^*(t) \sim v(t) \sim \mathcal{L}^{-1}\{ s^2 \mathcal{L}^2 (\vartheta(u) \}$$

In other words:

$$v(t) \sim \mathcal{L}^{-1}\{ s^2 \mathcal{L}^2 (\vartheta(u) \} \tag{9}$$

But as stated in ref. (1) formula [20], we have ("Farey dissection").

$$\mathcal{L}\{ \vartheta(u) \} = \frac{1}{s} \sum_{q=1}^{[\sqrt{n}]} \sum_{\substack{h=o \\ (h,q)=1}}^{q-1} \frac{\mu(q)}{\varphi(q)\ (s + 2\pi ih/q)} + A\, n^{\alpha+1/4+\varepsilon} \tag{10}$$

where:

$\mu(q)$ = Moebius function
$\varphi(q)$ = Euler's function
α = upper limit of the real part of the imaginary zeros of the L-series involved.

From [10] follows:

$$\mathcal{L}^2\{ \vartheta(u) \} \sim \frac{1}{s^2} \sum_{q_1} \sum_{\substack{h_1 \\ (h_1,q_1)=1}} \sum_{q_2} \sum_{\substack{h_2 \\ (h_2,q_2)=1}} \frac{\mu(q_1)\ \mu(q_2)}{\varphi(q_1)\ \varphi(q_2)\ (s+A_1)\ (s+A_2)}$$

where:

$$A_1 = 2\pi i h_1 / q_1 \qquad\qquad A_2 = 2\pi i h_2 / q_2$$

Next we separate the terms with $A_1 = A_2$ from the others where $A_1 \neq A_2$:
We obtain:

$$\mathcal{L}^2\{ \vartheta(u) \} \sim \frac{1}{s^2} \sum_{q} \sum_{\substack{h \\ (h,q)=1}} \frac{\mu^2(q)}{\varphi(q)^2\ \ (s + A)^2} + \tag{11}$$

$$+\ \frac{1}{s^2} \sum_{q_1} \sum_{h_1} \sum_{q_2} \sum_{h_2} \frac{\mu(q_1)\ \mu(q_2)}{\varphi(q_1)\ \varphi(q_2)\ (s+A_1)\ (s+A_2)}$$
$$A_1 \neq A_2$$

We insert now this expression in formula [9] obtaining thus:

$$\nu(t) \sim \mathcal{L}^{-1}\left\{ \sum_{q} \sum_{\substack{h \\ (h,q)=1}} \frac{\mu^2(q)}{\varphi(q)^2\ (s+A)^2} \right\} +$$

$$+\ \mathcal{L}^{-1}\left\{ \sum_{q_1} \sum_{\substack{h_1 \\ (h_1,q_1)=1}} \sum_{q_2} \sum_{\substack{h_2 \\ (h_2,q_2)=1}} \frac{\mu(q_1)\ \mu(q_2)}{\varphi(q_1)\ \varphi(q_2)\ (s+A_1)\ (s+A_2)} \right\}$$

As we are dealing with finite sums, we can write:

$$\nu(t) \sim \sum_{q} \sum_{\substack{h \\ (h,q)=1}} \left(\frac{\mu(q)}{\varphi(q)}\right)^2 \mathcal{L}^{-1}\left\{\frac{1}{(s+A)^2}\right\} +$$

$$+ \sum_{q_1} \sum_{\substack{h_1 \\ (h_1,q_1)=1}} \sum_{q_2} \sum_{\substack{h_2 \\ (h_2,q_2)=1}} \frac{\mu(q_1)\ \mu(q_2)}{\varphi(q_1)\ \varphi(q_2)}\ \mathcal{L}^{-1}\left\{\frac{1}{(s+A_1)\ (s+A_2)}\right\}$$

From elementary tables:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+A)^2}\right\} = te^{-At}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+A_1)\ (s+A_2)}\right\} = \frac{1}{A_2 - A_1}\left\{ e^{-A_1 t} - e^{-A_2 t} \right\}$$

Hence:

[12] $$\nu(t) \sim \sum_{q=1}^{[\sqrt{n}]} \sum_{h=o}^{q-1} \frac{\mu^2(q)}{\varphi^2(q)} e^{-At} t +$$

$$+ \sum_{q_1} \sum_{\substack{h_1 \\ (h_1,q_1)=1}} \sum_{q_2} \sum_{\substack{h_2 \\ (h_2,q_2)=1}} \frac{\mu(q_1)\ \mu(q_2)}{\varphi(q_1)\ \varphi(q_2)} \frac{e^{-A_1 t} - e^{-A_2 t}}{A_2 - A_1}$$

The term in the second line can be majorized as follows. The minimum value of $A_2 - A_1$ is $2\pi i(h_1/q_1 - h_2/q_2)$, where both fractions are contiguous Farey fractions.
It is known (ref. (2)) that:

$$\frac{h_1}{q_1} - \frac{h_2}{q_2} = o\left(\frac{1}{q_1^2}\right) = o\left(\frac{1}{q_2^2}\right)$$

so that

$$\left|\frac{1}{A_2 - A_1}\right| < o\left(\frac{1}{q_1^2}\right) = o\left(\frac{1}{q_2^2}\right)$$

Besides:

$$\left| e^{-A_1 t} - e^{-A_2 t} \right| < 2$$

Hence:

$$\left| \sum_{\substack{h_1=o \\ (h,q)=1}}^{q_1-1} \frac{e^{-A_1 t} - e^{-A_2 t}}{A_2 - A_1} \right| < \sum_{\substack{h_1=o \\ (h,q)=1}}^{q_1-1} \left| \frac{e^{-A_1 t} - e^{-A_2 t}}{A_2 - A_1} \right| < \varphi(q_1)\ O(q_1^2)$$

with an analogous result for the sum along h_2 .
It follows from the preceding inequalities:

$$\left| \sum_{q_1} \sum_{\substack{h_1 \\ (h_1,q_1)=1}} \sum_{q_2} \sum_{\substack{h_2 \\ (h_2,q_2)=1}} \frac{\mu(q_1)\ \mu(q_2)}{\varphi(q_1)\ \varphi(q_2)} \quad \frac{e^{-A_1 t} - e^{-A_2 t}}{A_2 - A_1} \right| <$$

$$\sum_{q_1=1}^{N} \sum_{q_2=1}^{N} O(q_1^2)\ O(q_2^2) = O(N^3)\ O(N^3) = O(N^6) \qquad (N = [\sqrt{n}])$$

so that [12] can be written as:

$$[13] \qquad \nu(t) \sim \sum_{q=1}^{N} \sum_{\substack{h=o \\ (h,q)=1}}^{q-1} \frac{\mu^2(q)}{\varphi^2(q)} \ e^{-At}\ t + O(N^6)$$

But:

$$\sum_{\substack{h=o \\ (h,q)=1}}^{q-1} e^{-At} = c_q(t) = \text{Ramanujan's sum}$$

Hence:

$$[14] \qquad \nu(t) \sim \sum_{q=1}^{N} \frac{\mu^2(q)}{\varphi^2(q)} \ c_q(t)\ t + O(N^6)$$

We choose now $N = [\sqrt[k]{t}]$ where k is a fixed number > 6.
We deduce from [14] that:

$$[15] \qquad \nu(t) \sim \sum_{q=1}^{[\sqrt[k]{t}]} \frac{\mu^2(q)}{\varphi^2(q)} \ c_q(t)\ t$$

in the cases where the series does not vanish.
Next, we have that:

$$\lim_{t\to\infty} \nu(t) = \lim_{t\to\infty} \sum_{q=1}^{[\sqrt[k]{t}]} \frac{\mu^2(q)}{\varphi^2(q)} \; c_q(t) \; t = \sum_{q=1}^{\infty} \frac{\mu^2(q)}{\varphi^2(q)} \; c_q(t) \; t$$

if the series at right is convergent.
According to R. Vaughan (ref. (2)) this is the actual case and its value is:

$$[16] \qquad \sum_{q=1}^{\infty} \frac{\mu^2(q)}{\varphi^2(q)} c_q(t) = 2 \prod_{p=3}^{\infty} \left\{ 1 - \frac{1}{(p-1)^2} \right\} \prod_{\substack{p/t \\ p\neq 2}} \frac{p-1}{p-2} = 1{,}3203 \ldots \prod_{\substack{p/t \\ p\neq 2}} \frac{p-1}{p-2}$$

if t is even, and zero if t is odd (ref. (3) p. 27).
Hence, we have proved the following result (valid for even t):

$$[17] \qquad \nu(t) \sim 2 \prod_{p=3}^{\infty} \left\{ 1 - \frac{1}{(p-1)^2} \right\} \prod_{p/t} \frac{p-1}{p-2} \; t$$

which is very exactly Conjecture [A] of Hardy-Littlewood in his paper ref. (3), which we reproduce textually here:

"**Conjecture [A] :** Every large even number is the sum of two primes. The asymptotic formula for the number of representatives is:

$$[4.11] \qquad N_2(n) \sim 2 \; C_2 \frac{n}{\log^2 n} \prod_{p/t} \left(\frac{p-1}{p-2} \right)$$

where p is an odd prime divisor of n , and

$$[4.12] \qquad C_2 = \prod_{\varpi=3}^{\infty} \left(1 - \frac{1}{(\varpi-1)^2} \right)$$ "

(In 4.12 ϖ denotes any prime number ≥ 2).

By Theorem C (ref. (3) p. 30), we have:

If n and r are of like parity, then

$$N_r(n) \sim \frac{\nu_r(n)}{(\log n)^r}$$

where N_r is the quantity of solutions of:

[18] $$n = p_1 + p_2 + \ldots + p_r$$

and:

$$\nu_r(n) = \sum \log p_1 \ \log p_2 \ \ldots \ \log p_n$$

where the sum is extended to the primes that fulfil [18].
Hardy and Littlewood state that r must be ≥ 3 ; but the proof they give also covers the case r=2. Hence, in connection with our [17].

$$N_2(t) \sim \frac{\nu(t)}{\log^2 t}$$

if t is even.
This is exactly (4.11) of Conjecture [A] , finally proved after 75 years.

5.- The same method can be used in order to prove the conditional theorems [B] and [D], and the conjectures [E], [G], [H], [I] and [J], of ref. (3).

This will be the subject of a next paper.

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(3) G. HARDY and J. E. LITTLEWOOD, *Some problems in "Partitio Numerorum" (III), On the expression of a number as a sum of primes*. Acta Math. (1923), 44, p. 1-70. The conjecture is stated in p. 32.

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