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THE IRRATIONALITY OF EULER'S CONSTANT (III)

by Aldo Peretti

1. Euler's constant

$$\gamma = \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln(n) \right) = 0{,}5772156649 \ \ldots$$

was first computed by Euler, using what is now known as the Euler-Maclaurin-sum formula, to 15 decimal places. By its importance, it is considered the third constant of Mathematics.

The computation of Euler's constant has not attracted the same public interest as calculating π , but it has still inspired the dedication of a few.

Very recently, Thomas Papanikolaou was able to compute one million of decimals of Euler's constant.

A recent paper (1) gives Vacca's series (given below, formula [1]) and repeats the well known assertion that the irrationality of γ remains open.

2. Vacca's series

In ref. (2) G. Vacca deduced, in 1910, that:

$$[1] \qquad \gamma = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} [\log_2 k]$$

where $[u]$ denotes the greatest integer function, and $\log_2 k$ is the logarithm of k in base 2 .
The author rediscovered this development in ref. (3).
The first terms of the series are:

$$[2] \qquad \gamma = \frac{1}{2} - \frac{1}{3} + \frac{2}{4} - \frac{2}{5} + \frac{2}{6} - \frac{2}{7} + \frac{3}{8} - \frac{3}{9} + \frac{3}{10} - \frac{3}{11} + \frac{3}{12} - \frac{3}{13} + + \frac{3}{14} + \frac{3}{15} + \frac{4}{16} + \ldots$$

The changes in the numerators take place when the denominator is a power of 2 .
Now we split the numerators in [2] as a sum of unities, and obtain:

$$[3] \quad \gamma = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{4} - \frac{1}{5} - \frac{1}{5} + \frac{1}{6} + \frac{1}{6} - \frac{1}{7} - \frac{1}{7} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} - \ldots$$

without affecting the convergence or the value of the series.

3. Euler's formula to transform a series into a continued fraction.

It runs as follows (ref. (4)):

$$[4] \qquad \frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_3} + \dots = \cfrac{1}{u_1 - \cfrac{u_1^2}{u_1 + u_2 - \cfrac{u_2^2}{u_2 + u_3 - \cfrac{u_3^2}{u_3 + u_4 - \cfrac{u_4^2}{\dots}}}}}$$

or, in Perron's notation:

$$[5] \qquad \frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_3} + \dots = \frac{1|}{|u_1} - \frac{u_1^2|}{|u_1 + u_2} - \frac{u_2^2|}{|u_2 + u_3} - \frac{u_3^2}{|u_3 + u_4} - \dots$$

Perron calls u_1^2, u_2^2. u_3^2, ... the "numerators", and u_1+u_2 , u_2+u_3 , u_3+u_4 ... the "denominators" of the continued fraction.

Each denominator u_n of the series at the left hand side of [4] (i.e.: the right hand side of [3]) takes the value $\pm m$ (m = natural number ≥ 2). Then the next term u_{n+1} takes the values $\pm m$ or $\pm (m + 1)$.

However, the sum $u_n + u_{n+1}$ can assume only four values, according to the following tree:

[6] $u_{n+1} + u_n =$

- $\pm m \quad \pm m$
  - $\to + 2\,m$ (a)
  - $\to - 2\,m$ (b)
- $\pm (m + 1) \quad \pm m$
  - $\to - 1$ (c)
  - $\to + 1$ (d)

Proof: There are only four possible alternatives in the sequence of the u_n in [3]:

(a) a denominator $+ m$ is followed by other denominator $+ m$. Then $u_n + u_{n+1} = 2\,m$.

(b) a denominator $- m$ is followed by other denominator $- m$. Then $u_n + u_{n+1} = - 2\,m$.

(c) a denominator $+ m$ is followed by other denominator $- (m+1)$. Then $u_n + u_{n+1} = - 1$.

(d) a denominator $- m$ is followed by other denominator $+ (m+1)$. . Then $u_n + u_{n+1} = + 1$.

The four remaining alternatives, $+m \ -m$; $-m \ +m$; $-(m+1) \ -m$; $+(m+1) \ +m$ are easily seen to be impossible.

According to [5], the "numerator" of a typical quotient is $u_n = m^2$, while the corresponding "denominator" $u_n + u_{n+1}$ takes some of the four values given in [6] .

Hence the successive quotients are of the form:

[7] $$\frac{-m^2|}{|\pm 1} \quad \text{or of the form} \quad \frac{-m^2|}{|\pm 2\,m}$$

Indeed, we have:

[8] $$\gamma = \cfrac{1}{2 - \cfrac{2^2}{-1 - \cfrac{3^2}{+1 - \cfrac{4^2}{8 - \cfrac{4^2}{-1 - \cfrac{5^2}{-10 - \cfrac{5^2}{1 - \cfrac{6^2}{12 - \cfrac{6^2}{\cdots}}}}}}}}}$$

The quotient $\frac{-m^2|}{|\pm 2\,m}$ can be simplified, but this operation requires of special care. We can write the quotient as:

[9] $$A - \cfrac{m^2}{\pm 2\,m - \cfrac{m^2}{B}}$$

(The alternative

$$A - \cfrac{m^2}{\pm 2\,m - \cfrac{(m+1)^2}{B}}$$

being impossible due to [6] (a) and (b)).

where A denotes the span before m^2 , and B the remainder terms of the fraction. It is now evident that:

[10] $$A - \cfrac{m^2}{\pm 2\,m - \cfrac{m^2}{B}} \quad = \quad A - \cfrac{m}{\pm 2 - \cfrac{m}{B}}$$

Still it can occur that m be an even number $m = 2q$, in whose case the right hand side of [10] takes the form:

$$A - \cfrac{2q}{\pm 2 - \cfrac{2q}{B}}$$

where again we can simplify the number 2, obtaining finally:

$$A - \cfrac{q}{\pm 1 - \cfrac{q}{B}}$$

Summaryzing, in this last case, we can replace the quotient

$$\frac{-m^2|}{|\pm m} \qquad \text{by} \qquad \frac{-q|}{|\pm 1}$$

and the important conclusion of this paragraph is:

The successive quotients of the simplified continued fraction [8], (which we shall denotes as SCF8) are of the types:

$$\frac{-m^2|}{|\pm 1} \quad ; \quad \frac{-m|}{\pm 2} \ \text{(m odd)} \quad ; \quad \frac{-q|}{\pm 1} \ \text{(m even} = 2q)$$

4. At this stage we use the following theorem (taken from ref. (4) Ch. XXXI, § 446) :

If each quotient of the fraction:

$$\cfrac{b_1}{a_1 - \cfrac{b_2}{a_2 - \cfrac{b_3}{a_3 - \dots}}} = \frac{b_1|}{|a_1} - \frac{b_2|}{|a_2} - \frac{b_3|}{|a_3} - \dots$$

is a proper (irreducible) fraction, with integer "numerators" and "denominators", and if the value of the infinite continued fraction that begins with any component a_n or b_n is less than unity, the fraction represents an irrational number.

Now, the whole continued fraction is less than unity, because $\gamma < 1$, and the infinite continued fraction that begins with any component reflects merely the value of series [2] from a certain term onwards.

This last value cannot exceed (in modulus), the value of series [1] from any term onwards. But this quantity, due to the alternancy of the signs, cannot exceed $[\log_2 k] / k$, which, for every k , is < 1 .

Hence, SCF8 fulfils with all the requirements of the theorem, and is, consequently, an irrational number.

5. Other possibility is to use the identity (also due to Euler):

$$[11] \quad \frac{1}{u_1} - \frac{1}{u_2} + \frac{1}{u_3} - \dots + \frac{(-1)^{n-1}}{u_n} = \cfrac{1}{u_1 + \cfrac{u_1^2}{u_2 - u_1 + \cfrac{u_2^2}{u_3 - u_2 + \dots \cfrac{}{+ \cfrac{u_{n-1}^2}{u_n - u_{n-1}}}}}}$$

that gives us the development:

$$\gamma = \cfrac{1}{2 + \cfrac{2^2}{1 + \cfrac{3^2}{1 + \cfrac{4^2}{-8 + \cfrac{4^2}{-1 + \cfrac{5^2}{10 + \cfrac{5^2}{\dots}}}}}}}$$

whose analogy with [8] is evident.
In fact, we find that appear the same numbers, but with other signs.
This development admits the same simplifications than [8], and is of the type:

$$\cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \dots}}} = \frac{b_1|}{|a_1} + \frac{b_2|}{|a_2} + \frac{b_3|}{|a_3} + \dots$$

In the fractions of this type it is sufficient condition that $(b_n , a_n) = 1$ (with b_n and a_n integers), in order that the quantity be irrational (ref. (4), § 445).

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