NNTDM 3 (1997) 4, 181 - 184

A relation of modular discriminant $\Delta(\tau)$

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ABSTRACT. Let $\Delta(\Lambda_\tau) = \Delta(\tau)$ be modular discriminant and $\Omega(\tau) = (2\pi)^4\eta(\tau)^8$, where $\eta(\tau)$ be Dedekind η-function.

(a) $\Delta(\tau) = \pm\frac{1}{16}\Omega(\frac{\tau+1}{2})\Omega(\frac{\tau}{2})(\bar{\rho}\Omega(\frac{\tau+1}{2}) + \rho\Omega(\frac{\tau}{2}))$.

(b) $\Delta(\tau) = \pm\left(\frac{1}{16}\Omega(\frac{\tau+1}{2})^2\Omega(\frac{\tau}{2})^2 - 16\frac{h_1(\tau)^2}{\Omega(\frac{\tau}{2})\Omega(\frac{\tau+1}{2})}\right)$. (Theorem 2.1)

§1. Preliminaries

We will use the term ***elliptic curve*** to mean an abelian variety of dimension 1, or, what is the same, an irreducible non-singular projective algebraic curve of genus 1 furnished with a point O, the origin for the group law.

It is standard to set

$$g_2 = g_2(\Lambda) = 60G_4 \quad \text{and} \quad g_3 = g_3(\Lambda) = 140G_6.$$

Then the algebraic relation between $\mathfrak{p}(z)$ and $\mathfrak{p}'(z)$ is

$$\mathfrak{p}'(z)^2 = 4\mathfrak{p}(z)^3 - g_2\mathfrak{p}(z) - g_3.$$

Let g_2 and g_3 be the quantities associated to a lattice $\Lambda \subset \mathbb{C}$. Its discriminant

$$\Delta(\Lambda) = g_2^3 - 27g_3^2$$

is not zero. We called $\Delta(\Lambda_\tau) = \Delta(\tau)$ the *modular discriminant.*

The *Dedekind η-function* $\eta(\tau)$ is defined by the product

$$\eta(\tau) = e^{(2\pi i\tau)/24}\prod_{n=1}^{\infty}(1-q^n) \quad \text{for } \tau \in \mathfrak{h}, q = e^{2\pi i\tau}.$$

Let x and y be relatively prime integers with $y > 0$.

The *Dedekind sum* $s(x, y)$ is defined to be

$$s(x,y) = \sum_{j=1}^{y-1}\frac{j}{y}\left(\frac{jx}{y} - \left[\frac{jx}{y}\right] - \frac{1}{2}\right).$$

(The square bracket denotes the greatest integer function.)

1991 AMS Subject Classification: 11GXX, 11F27

keywords and phrases :modular discriminant, theta series, Dedekind eta-function

Proposition 1.1.[14].

(a) *The Dedekind η-function satiesfies the identities*

$$\eta(\tau+1) = e^{\frac{2\pi i}{24}}\eta(\tau), \quad \text{and } \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau).$$

Here we take the branch of $\sqrt{\ }$ which is positive on the positive real axis.

(b)

$$\Delta(\tau) = (2\pi)^{12}\eta(\tau)^{24}.$$

Proposition 1.2[7]. *Let $\Omega(\tau) = (2\pi)^4\eta(\tau)^8$, where $\eta(\tau)$ be Dedekind η-function and let $\rho = e^{\frac{2\pi i}{3}}$ and $-\bar{\rho} = e^{\frac{2\pi i}{6}}$.*

(a)
$$\eta(\tau)^{48} = \frac{1}{256}\Big[2\eta(\tfrac{\tau}{2})^{24}\eta(\tfrac{\tau+1}{2})^{24} + \eta(\tfrac{\tau}{2})^{16}\eta(\tfrac{\tau+1}{2})^{16}\left(\rho\eta(\tfrac{\tau+1}{2})^{16} + \bar{\rho}\eta(\tfrac{\tau}{2})^{16}\right)\Big].$$

(b)
$$\Omega(\tau)^6 = \frac{1}{256}\Omega(\tfrac{\tau}{2})^2\Omega(\tfrac{\tau+1}{2})^2\left[\Omega(\tfrac{\tau+1}{2})\left(\Omega(\tfrac{\tau}{2}) + \rho\Omega(\tfrac{\tau+1}{2})\right) + \Omega(\tfrac{\tau}{2})\left(\Omega(\tfrac{\tau+1}{2}) + \bar{\rho}\Omega(\tfrac{\tau}{2})\right)\right].$$

§2.Main theorem

Theorem 2.1. *Let*

$$h_r(\tau)^{2n} = \frac{1}{(256)^n}\left[2\Omega(\frac{\tau}{2})^3\Omega(\frac{\tau+1}{2})^3 + \Omega(\frac{\tau}{2})^2\Omega(\frac{\tau+1}{2})^2 \cdot \left(\rho\Omega(\frac{\tau+1}{2})^r + \bar{\rho}\Omega(\frac{\tau}{2})^r\right)\right]^n,$$

for arbitrary n.

(a) $\Delta(\tau) = \pm\dfrac{1}{16}\Omega(\tfrac{\tau+1}{2})\Omega(\tfrac{\tau}{2})(\bar{\rho}\Omega(\tfrac{\tau+1}{2}) + \rho\Omega(\tfrac{\tau}{2}))$.

(b) $\Delta(\tau) = \pm\left(\dfrac{1}{16}\Omega(\tfrac{\tau+1}{2})^2\Omega(\tfrac{\tau}{2})^2 - 16\dfrac{h_1(\tau)^2}{\Omega(\tfrac{\tau}{2})\Omega(\tfrac{\tau+1}{2})}\right)$.

Proof.

(a) By definition of modular discriminant and Proposition 1.2,

$$\begin{aligned}
\Delta(\tau)^2 &= \frac{\Omega(\frac{\tau+1}{2})^2\Omega(\frac{\tau}{2})^2}{256}[2\Omega(\frac{\tau+1}{2})\Omega(\frac{\tau}{2}) + \rho\Omega(\frac{\tau+1}{2})^2 + \bar{\rho}\Omega(\frac{\tau}{2})^2] \\
&= \frac{\Omega(\frac{\tau+1}{2})^2\Omega(\frac{\tau}{2})^2}{256}[2\Omega(\frac{\tau+1}{2})\Omega(\frac{\tau}{2}) + \bar{\rho}^2\Omega(\frac{\tau+1}{2})^2 + \rho^2\Omega(\frac{\tau}{2})^2] \\
&= \frac{\Omega(\frac{\tau+1}{2})^2\Omega(\frac{\tau}{2})^2}{256}[2\bar{\rho}\rho\Omega(\frac{\tau+1}{2})\Omega(\frac{\tau}{2}) + \bar{\rho}^2\Omega(\frac{\tau+1}{2})^2 + \rho^2\Omega(\frac{\tau}{2})^2] \\
&= \frac{\Omega(\frac{\tau+1}{2})^2\Omega(\frac{\tau}{2})^2}{256}[\bar{\rho}\Omega(\frac{\tau+1}{2}) + \rho\Omega(\frac{\tau}{2})]^2 \\
&= \left[\frac{1}{16}\Omega(\frac{\tau+1}{2})\Omega(\frac{\tau}{2})(\bar{\rho}\Omega(\frac{\tau+1}{2}) + \rho\Omega(\frac{\tau}{2}))\right]^2.
\end{aligned}$$

Therefore, $\Delta(\tau) = \frac{1}{16}\Omega(\frac{\tau+1}{2})\Omega(\frac{\tau}{2})(\bar{\rho}\Omega(\frac{\tau+1}{2}) + \rho\Omega(\frac{\tau}{2}))$.

(b) By (a),

$$\begin{aligned}
\Delta(\tau) &= \frac{1}{16}\Omega(\frac{\tau+1}{2})\Omega(\frac{\tau}{2})(\bar{\rho}\Omega(\frac{\tau+1}{2}) + \rho\Omega(\frac{\tau}{2})) \\
&= \frac{1}{16}\Omega(\frac{\tau+1}{2})\Omega(\frac{\tau}{2})(\rho^2\Omega(\frac{\tau+1}{2}) + \bar{\rho}^2\Omega(\frac{\tau}{2})) \\
&= \frac{1}{16}\Omega(\frac{\tau+1}{2})\Omega(\frac{\tau}{2})((-\rho-1)\Omega(\frac{\tau+1}{2}) + (-\bar{\rho}-1)\Omega(\frac{\tau}{2})) \\
&= \frac{1}{16}\Omega(\frac{\tau+1}{2})\Omega(\frac{\tau}{2})(-(\rho\Omega(\frac{\tau+1}{2}) + \bar{\rho}\Omega(\frac{\tau}{2})) - (\Omega(\frac{\tau+1}{2}) + \Omega(\frac{\tau}{2}))).
\end{aligned}$$

Thus,

$$\rho\Omega(\frac{\tau+1}{2}) + \bar{\rho}\Omega(\frac{\tau}{2}) = -16\frac{\Delta(\tau)}{\Omega(\frac{\tau+1}{2})\Omega(\frac{\tau}{2})} - \Omega(\frac{\tau+1}{2})\Omega(\frac{\tau}{2}).$$

Also by definition of $h_1(\tau)^2$,

$$\begin{aligned}
h_1(\tau)^2 &= \frac{\Omega(\frac{\tau+1}{2})^2\Omega(\frac{\tau}{2})^2}{256}\left[2\Omega(\frac{\tau+1}{2})\Omega(\frac{\tau}{2}) + \rho\Omega(\frac{\tau+1}{2}) + \bar{\rho}\Omega(\frac{\tau}{2})\right] \\
&= \frac{\Omega(\frac{\tau+1}{2})^2\Omega(\frac{\tau}{2})^2}{256}\left[2\Omega(\frac{\tau+1}{2})\Omega(\frac{\tau}{2}) - \frac{16\Delta(\tau)}{\Omega(\frac{\tau+1}{2})\Omega(\frac{\tau}{2})} - \Omega(\frac{\tau+1}{2})\Omega(\frac{\tau}{2})\right] \\
&= \frac{\Omega(\frac{\tau+1}{2})^2\Omega(\frac{\tau}{2})^2}{256}\left[\Omega(\frac{\tau+1}{2})\Omega(\frac{\tau}{2}) - \frac{16\Delta(\tau)}{\Omega(\frac{\tau+1}{2})\Omega(\frac{\tau}{2})}\right] \\
&= \frac{\Omega(\frac{\tau+1}{2})^3\Omega(\frac{\tau}{2})^3}{256} - \frac{\Omega(\frac{\tau+1}{2})\Omega(\frac{\tau}{2})\Delta(\tau)}{16}.
\end{aligned}$$

Thus,

$$h_1(\tau)^2 - \frac{\Omega(\frac{\tau+1}{2})^3\Omega(\frac{\tau}{2})^3}{256} = -\frac{\Omega(\frac{\tau+1}{2})\Omega(\frac{\tau}{2})\Delta(\tau)}{16}.$$

Therefore,

$$\Delta(\tau) = \frac{1}{16}\Omega(\frac{\tau+1}{2})^2\Omega(\frac{\tau}{2})^2 - 16\frac{h_1(\tau)^2}{\Omega(\frac{\tau}{2})\Omega(\frac{\tau+1}{2})}. \qquad \square$$

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