NNTDM 3 (1997) 4, 179 - 180

ONE EXTREMAL PROBLEM. 7

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Devoted to Prof. Aldo Peretti
for his 65-th anniversary

The following problem is a combination of the ideas from [1-3]: to determine the value for K for which $m_K = \max_{0 \le k \le N} m_k$, where $n \ge 1$ is a fixed natural number $N = [\frac{n+1}{2}]$ and $m_k = \binom{n-k}{k}.n^k$.

It is directly checked that $m_0 = 1 < m_1 = (n - 1).n$ and

$m_{N-1} = \binom{n-N+1}{N-1}.n^{N-1} > m_N = \binom{n-N}{N}.n^N$.

Therefore, it exists at least one k $(0 \le k \le N)$ for which:

$$\begin{array}{l} m_{k-1} < m_k \\ m_{k+1} \le m_k \end{array} \qquad (1)$$

Let us assume the existence of a natural number q $(1 \le q \le N)$ for which:

$$\begin{array}{l} m_{q-1} \ge m_q \\ m_{q+1} \ge m_q \end{array} \qquad (2)$$

Hence:

$$\frac{(n-q+1).(n-q)...(n-2.q+3)}{(q-1)!}.n^{q-1} \ge \frac{(n-q).(n-q-1)...(n-2.q+1)}{q!}.n^q$$

$$\frac{(n-q-1).(n-q-2)...(n-2.q-1)}{(q+1)!}.n^{q+1} \ge \frac{(n-q).(n-q-1)...(n-2.q+1)}{q!}.n^q$$

i.e., the following two inequalities are valid simultaneously:

$$(n - q + 1).q \ge (n - 2.q + 2).(n - 2.q + 1).n$$

$$(n - 2.q).(n - 2.q - 1).n \ge (n - q).(q + 1).$$

Hence

$(n - q + 1).q \ge (n - 2.q + 2).(n - 2.q + 1).n$

$= (n - 2.q).(n - 2.q - 1).n + 4.n^2 - 8.n.q + 5.n$

$> (n - 2.q).(n - 2.q - 1).n + 1 \ge (n - q).(q + 1) + 1$

$\ge (n - q + 1).q$

which is impossible.

Therefore, there are not three numbers for which (2) is valid. In particular, there are not three numbers for which $m_{q-1} = m_q = m_{q+1}$.

The following question is interesting, too: is there a natural number k $(1 \le k \le N)$ for which

$$m_{k+1} = m_k \ ? \qquad (3)$$

Let k have this property. Then

$$\frac{(n-k-1).(n-k-2)...(n-2.k-1)}{(k+1)!}.n^{k+1} = \frac{(n-k).(n-k-1)...(n-2.k+1)}{k!}.n^k$$

i.e.

$(n - 2.k).(n - 2.k - 1).n = (n - k).(k + 1).$

Then

$(4.n + 1).k^2 - (4.n^2 - n - 1).k + n^3 - n^2 - n = 0$

and

$$k = \frac{4.n^2 - n - 1 \pm 4.n^3 + 13.n^2 + 6.n + 1}{2.(4.n + 1)}.$$

From $k \leq N$ follows that the sign "$\pm$" must be changed with the sign "$-$". Therefore, the equality (3) will be valid only for these n for which $\frac{4.n^2 - n - 1 - 4.n^3 + 13.n^2 + 6.n + 1}{2.(4.n + 1)}$ is a natural number. The smallest n for which there are two numbers m_k and m_{k+1} for which (3) is valid is $n = 6$. Then $m_2 = m_3 = 216$.

Now, we shall show the solutions of (1). From that it follows that

$$\frac{(n-k+1).(n-k)...(n-2.k+3)}{(k-1)!}.n^{k-1} < \frac{(n-k).(n-k-1)...(n-2.k+1)}{k!}.n^k$$

$$\frac{(n-k-1).(n-k-2)...(n-2.k-1)}{(k+1)!}.n^{k+1} \leq \frac{(n-k).(n-k-1)...(n-2.k+1)}{k!}.n^k$$

i.e., the following two inequalities are valid simultaneously:

$$(n - k + 1).k < (n - 2.k + 2).(n - 2.k + 1).n$$

$$(n - 2.k).(n - 2.k - 1).n \leq (n - k).(k + 1)$$

or

$$(4.n + 1).k^2 - (4.n^2 + 7.n + 1).k + n^3 + 3.n^2 + 2.n > 0$$

$$(4.n + 1).k^2 - (4.n^2 - n - 1).k + n^3 - n^2 - n \leq 0$$

Therefore every solution k satisfies the condition

$$k \in ([-\infty, \frac{4.n^2 + 7.n + 1 - \sqrt{D}}{2.(4.n + 1)}] \cup [\frac{4.n^2 + 7.n + 1 + \sqrt{D}}{2.(4.n + 1)}, +\infty]) \cap$$

$$[\frac{4.n^2 - n - 1 - \sqrt{E}}{2.(4.n + 1)}, \frac{4.n^2 - n - 1 - \sqrt{E}}{2.(4.n + 1)}]$$

i.e.

$$k \in [\frac{4.n^2 - n - 1 - \sqrt{E}}{2.(4.n + 1)}, \frac{4.n^2 + 7.n + 1 - \sqrt{D}}{2.(4.n + 1)}] \cup$$

$$[\frac{4.n^2 + 7.n + 1 + \sqrt{D}}{2.(4.n + 1)}, \frac{4.n^2 - n - 1 - \sqrt{E}}{2.(4.n + 1)}],$$

where $D = 4.n^3 - 13.n^2 + 6.n + 1$ and $E = 4.n^4 + 5.n^2 - 2.n + 1$.

But $[\frac{4.n^2 + 7.n + 1 + \sqrt{D}}{2.(4.n + 1)}, \frac{4.n^2 - n - 1 - \sqrt{E}}{2.(4.n + 1)}] = \emptyset$

and

$$\frac{4.n^2 + 7.n + 1 - \sqrt{D}}{2.(4.n + 1)} - \frac{4.n^2 - n - 1 - \sqrt{E}}{2.(4.n + 1)} = 1 + \frac{\sqrt{E} - \sqrt{D}}{2.(4.n + 1)};$$

and

$$0 < \frac{\sqrt{E} - \sqrt{D}}{2.(4.n + 1)} < 1.$$

Hence, the unique solution (the unique two solutions) of (1) is this k (are these k and $k + 1$) for which

$$\frac{4.n^2 - n - 1 - \sqrt{E}}{2.(4.n + 1)} \leq k < \frac{4.n^2 + 7.n + 1 - \sqrt{D}}{2.(4.n + 1)}.$$

Therefore (as in [1]):

$$m_0 < m_1 ... < m_{k-1} < m_k \geq m_{k+1} > m_{k+2} ... > m_N.$$

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[1] Atanassov K. One extremal problem., Bull. of Number Theory and Related Topics Vol. VIII (1984), No. 3, 6-12.

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[3] Atanassov K. One extremal problem. 4., Bull. of Number Theory and Related Topics Vol. XI (1987), No. 1, 64-71.