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A MODIFICATION OF A. MULLIN'S INEQUALITY

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ABSTRACT A modification of one A. Mullin's hypothesis is introduced and solved.

In [1] A. Mullin proposed several hypotheses, one of which (Problem 1) the author proved in [2]. He essentialy improved (and reduced) the proof in [3] and modified the Mullin's hypothesis in [4].

Here we shall introduce a further modification of it.

First, following [3,4] we shall define for the natural number $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, where k, α_1, α_2, ... $\alpha_k \geq 1$ are natural numbers and $p_1, p_2, ..., p_k$ are different prime numbers:

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}.(p_i - 1),$$

$$\psi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}.(p_i + 1),$$

$$set(n) = \{p_1, p_2, ..., p_k\},$$

$$cas(n) = k,$$

$$mult(n) = \prod_{i=1}^{k} p_i,$$

$$sum_2(n) = \sum_{i=1}^{k} p_i^2,$$

$$dim(n) = \sum_{i=1}^{k} \alpha_i.$$

The following assertion is valid.

THEOREM: For every natural number n, such that $dim(n) \geq 2$:

$$\varphi(n).\psi(n) \leq n^2 - (cas(n) - 2).mult(n) - (dim(n) - 1).sum_2(n) + 1.$$

Proof: Let $dim(n) = 2$. Then:
(a) $n = p^2$ for some prime number p and therefore $cas(n) = 1$ and:

$$n^2 + mult(n) - sum_2(n) - \varphi(n).\psi(n) + 1 = p^4 + p - p^2 - p.(p-1).p.(p+1) + 1 = p + 1 > 0;$$

(b) $n = pq$ for some prime numbers p and q and therefore $cas(n) = 2$ and:

$$n^2 - sum_2(n) - \varphi(n).\psi(n) + 1 = p^2.q^2 - p.q - (p-1).(q-1).(p+1).(q+1) + 1 = 0,$$

i.e., the assertion is valid for $dim(n) = 2$.

Now, let us assume that the assertion is valid for all natural numbers n such that $dim(n) \leq m$ for some natural number $m \geq 2$. We shall prove that the assertion is valid for the numbers n' with $dim(n') = m + 1$, too.

Let p be a prime number. We shall construct the number $n' = n.p$ for which $dim(n') = dim(n) + 1 = m + 1$ and we shall prove that the assertion is valid for n'.

Let

$$A \equiv n'^2 - (cas(n') - 2).mult(n') - (dim(n') - 1).sum_2(n') + 1 - \varphi(n').\psi(n').$$

There are two possibilities for p and n:
(a) if $p \not\in set(n)$, then:
(a.1) if $cas(n) = 1$, i.e., $n = q^m$ for some prime number q and $cas(n') = 2$, then by induction:

$$A = p^2.q^{2m} - m.(p^2 + q^2) + 1 - (p^2 - 1).q^{2m-2}.(q^2 - 1)$$

$$\geq p^2.((m-1).q^2 - 1 + q^{2m-2}.(q^2 - 1)) - m.(p^2 + q^2) + 1 - (p^2 - 1).q^{2m-2}.(q^2 - 1)$$
$$= (m-1).p^2.q^2 - p^2 + p^2.q^{2m-2}.(q^2 - 1) - m.(p^2 + q^2) + 1 - (p^2 - 1).q^{2m-2}.(q^2 - 1)$$
$$= (m-1).p^2.q^2 - p^2 - m.(p^2 + q^2) + 1 + q^{2m-2}.(q^2 - 1) > 0$$

for $m \geq 2$.
(a.2) if $cas(n) = 2$, the inequality is proved analogically.
(a.3) if $cas(n) \geq 3$, then $sum_2(n) \geq 2^2 + 3^2 + 5^2 = 38$ and by induction:

$$A = n^2.p^2 - (cas(n) - 1).mult(n).p - dim(n).(sum_2(n) + p^2) + 1 - \varphi(n).\psi(n).(p^2 - 1)$$

$$= p^2.(\varphi(n).\psi(n) + (cas(n) - 2).mult(n) + (dim(n) - 1).sum_2(n) - 1)$$
$$-(cas(n) - 1).mult(n).p - dim(n).(sum_2(n) + p^2) + 1 - \varphi(n).\psi(n).(p^2 - 1)$$

$$> p.mult(n).((cas(n) - 2).p - cas(n) + 1) + \frac{1}{2}.(p^2.(dim(n) - 1).sum_2(n) - 2.dim(n).sum_2(n))$$

$$+\frac{1}{2}.(p^2.(dim(n) - 1).sum_2(n) - 2.p^2.(dim(n) + 1)) + 1 > 0,$$

because, easily it can be seen that every one of the three terms is a positive number.
(b) if $p \in set(n)$, then $cas(n') = cas(n), sum_2(n') = sun_2(n)$ and (b.1) if $cas(n) = 1$, i.e., $n = p^m$ then:

$$A = p^{2m+2} + p - m.p^2 + 1 - p^{2m}.(p^2 - 1) = p^{2m} + p - m.p^2 + 1 > 0.$$

(b.2) if $cas(n) = 2$, the inequality is proved analogically.
(b.3) if $cas(n) \geq 3$, then $sum_2(n) \geq 2^2 + 3^2 + 5^2 = 38$ and

$$A = n^2.p^2 - (cas(n) - 2).mult(n) - dim(n).sum_2(n) + 1 - p^2.\varphi(n).\psi(n)$$

$$= p^2.(\varphi(n).\psi(n) + (cas(n) - 2).mult(n) + (dim(n) - 1).sum_2(n) - 1)$$

$$-(cas(n) - 2).mult(n) - dim(n).sum_2(n) + 1 - p^2.\varphi(n).\psi(n)$$

$$= (p^2 - 1).(cas(n) - 2).mult(n)) + p^2.(dim(n) - 1).sum_2(n) - 1) - dim(n).sum_2(n) + 1 > 0,$$

i.e., the assertion is valid for $dim(n') = m + 1$ which completes the proof of the theorem.

REFERENCES

[1] Mullin A., Problems, Bulletin of Number Theory and Related Topics Vol. VI (1982), No. 3, 218-219.

[2] Atanassov K., On one problem of A. Mullin. Bulletin of Number Theory and Related Topics Vol. VIII (1984), No. 3, 1-5.

[3] Atanassov K. Short proof of a hypothesis of A. Mullin. Bulletin of Number Theory and Related Topics Vol. IX (1985), No. 2, 9-11.

[4] Atanassov K. Some properties of φ and σ functions. Comptes Rendus de l'Academie bulgare des Sciences, Tome 40, 1987, No. 4, 35-38.