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ON SOME ANALOGUES OF THE BOURQUE-LIGH CONJECTURE on LCM matrices

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Abstract

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers. The $n \times n$ matrix (S) whose i, j-entry is the greatest common divisor (x_i, x_j) of x_i and x_j is called the GCD matrix on S. The LCM matrix $[S]$ on S is defined analogously. It is a direct consequence of a known determinant evaluation that the GCD matrix is always nonsingular on gcd-closed sets. Bourque and Ligh conjectured that the LCM matrix is always nonsingular on gcd-closed sets. It has been shown that this conjecture does not hold. In this paper we study certain analogues of this conjecture relating to GCD and LCM matrices on lcm-closed sets and some unitary analogues of GCD and LCM matrices.

1. Introduction

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers. The $n \times n$ matrix (S) whose i, j-entry is the greatest common divisor (x_i, x_j) of x_i and x_j is called the GCD matrix on S (see [2]). The LCM matrix $[S]$ on S is defined analogously (see [1]). The set S is said to be factor-closed if $d \in S$ whenever $x_i \in S$ and $d|x_i$. The set S is said to be gcd-closed if $(x_i, x_j) \in S$ whenever $x_i, x_j \in S$ (see [3]). Note that if S is factor-closed, then S is gcd-closed. For a list of papers on GCD and LCM matrices, see [6] and [7].

H. J. S. Smith [8] showed that if S is factor-closed, then

$$\det(S) = \prod_{k=1}^{n} \phi(x_k) \tag{1}$$

and

$$\det[S] = \prod_{k=1}^{n} \pi(x_k)\phi(x_k), \tag{2}$$

where ϕ is Euler's totient function and π is the multiplicative function such that $\pi(p^a) = -p$ for all prime powers p^a (> 1). The function ϕ is always positive and the function π never assumes the value zero. Therefore the GCD matrix (S) and the LCM matrix $[S]$ are nonsingular on factor-closed sets.

Beslin and Ligh [3, Theorem 1] showed that if S is gcd-closed, then

$$\det(S) = \prod_{k=1}^{n} \sum_{d} \phi(d), \tag{3}$$

where for each k, d runs through the divisors of x_k such that $d \not| \, x_t$ for $x_t < x_k$. Note that if S is factor-closed, then (3) reduces to (1). Since $\phi(d) > 0$ for all d, it is clear that the GCD matrix (S) is nonsingular on gcd-closed sets.

Bourque and Ligh [4, Theorem 5] showed that if S is gcd-closed, then

$$\det[S] = \prod_{k=1}^{n} x_k^2 \sum_{d} \pi(d)\phi(d)/d^2, \tag{4}$$

where the sum over d is as in (3). Note that if S is factor-closed, then (4) reduces to (2). Bourque and Ligh [4] also *conjectured* that the LCM matrix $[S]$ is always nonsingular on gcd-closed sets. In [7] we showed that this conjecture does not hold. We showed that $S = \{1, 2, 3, 4, 5, 6, 10, 45, 180\}$ is a gcd-closed set such that $\det[S] = 0$. The sum in (4) is zero for $x_n = 180$. Recently Shen Ze Chun [5] made a more general conjecture: If S is gcd-closed and $r \neq 0$, then the LCM power matrix $[S^r]$ is nonsingular, where $[S^r]$ is the $n \times n$ matrix whose i,j-entry is $[x_i, x_j]^r$. Our example shows that this conjecture does not hold for $r = 1$.

In [6] we introduced the concept of an lcm-closed set and gave evaluations for $\det(S)$ and $\det[S]$ on lcm-closed sets. In [6] we also evaluated the determinants of certain unitary analogues of the GCD and LCM matrices. These evaluations are analogous to the evaluations given in (3) and (4) and give rise to the study of analogues of the Bourque-Ligh [4] conjecture, that is, to the study of nonsingularity of these matrices of [6]. In Theorems 3-9 we give positive or negative answers to the nonsingularity of these matrices.

2. Preliminaries

We assume that the reader is familiar with the basic concepts in number theory. In this section we review some more specialized material (see e.g. [6]).

The set $S = \{x_1, x_2, \ldots, x_n\}$ of distinct positive integers is said to be lcm-closed if $[x_i, x_j] \in S$ for every $x_i, x_j \in S$. If S is lcm-closed, then $x_i \mid \max S$ for all $x_i \in S$. Namely, otherwise $[x_i, \max S] > \max S$. Throughout this paper we set $x_n = \max S$ and denote $x_n = \max S = m$.

The reciprocal set $S^{(-1)}$ of S is defined by $S^{(-1)} = \{\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\}$. The product of m and $S^{(-1)}$ is defined as the set $mS^{(-1)} = \{\frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n}(=1)\}$. It can be shown that if S is lcm-closed, then $mS^{(-1)}$ is gcd-closed.

Let r and s be positive integers. A positive divisor d of r is said to be a unitary divisor of r if $(d, r/d) = 1$. If d is a unitary divisor of r, we write $d\|r$. The greatest common unitary divisor (gcud) of r and s is denoted by $(r, s)^{**}$. The symbol $(r, s)^*$ is used to denote the greatest unitary divisor of s which is a divisor of r. We refer to $(r, s)^*$ as the semi-unitary greatest common divisor (sugcd) of r and s. Note that there is some inconsistency in the notations $(r, s)^{**}$ and $(r, s)^*$ in the literature. The least common unitary multiple (lcum) is defined by

$$[r, s]^{**} = \frac{rs}{(r, s)^{**}}.$$

The unitary convolution of two arithmetical functions f and g is defined by

$$(f \oplus g)(r) = \sum_{d\|r} f(d)g(r/d).$$

The unitary analogue μ^* of the Möbius function is the inverse of constant function 1 under the unitary convolution. It is well known that μ^* is the multiplicative function such that $\mu^*(p^a) = -1$ for all prime powers p^a (> 1). The unitary analogue $\phi^*(r)$ of the Euler totient function is defined as the number of integers $x \pmod r$ such that $(x, r)^* = 1$. It is well known that $\phi^* = I \oplus \mu^*$, where $I(r) = r$ for all r.

The set S is said to be ud-closed if all unitary divisors of any member of S belong to S. The set S is said to be gcud-closed if $(x_i, x_j)^{**} \in S$ for every $x_i, x_j \in S$, and S is said to be sugcd-closed if $(x_i, x_j)^* \in S$ for every $x_i, x_j \in S$.

We denote by $(S)^*$ the $n \times n$ matrix whose i, j-entry is $(x_i, x_j)^*$. The matrices $(S)^{**}$ and $[S]^*$ are defined analogously. We refer to the matrices $(S)^*$, $(S)^{**}$ and $[S]^*$ as the SUGCD matrix, the GCUD matrix and the LCUM matrix, respectively.

3. The results

Theorems 1 and 2 were already noted in Section 1, see (3) and (4). Theorem 2 is the negative answer to the Bourque-Ligh conjecture.

Theorem 1 *The GCD matrix (S) on a gcd-closed set S is always nonsingular.*

Theorem 2 *The LCM matrix $[S]$ on a gcd-closed set S is not always nonsingular.*

We now present certain analogues of these theorems.

Theorem 3 *The GCD matrix (S) on an lcm-closed set S is always nonsingular.*

Proof It is known [6, Theorem 3.3 with $f = I$] that

$$\det(S) = m^{-n} \prod_{k=1}^{n} x_k^2 \sum_{d} \phi(d), \tag{5}$$

where for each k, d runs through the divisors of $\frac{m}{x_k}$ such that $d \nmid \frac{m}{x_t}$ for $x_k < x_t$. Since $\phi(d) > 0$ for all d, the GCD matrix (S) is nonsingular on lcm-closed sets. 2

Theorem 4 *The LCM matrix $[S]$ on an lcm-closed set S is not always nonsingular.*

Proof It is known [6, Theorem 3.4 with $f = I$] that

$$\det[S] = m^n \prod_{k=1}^{n} \sum_{d} \pi(d)\phi(d)/d^2, \tag{6}$$

where the sum over d is as in (5). We recall that if S is lcm-closed, then $mS^{(-1)}$ is gcd-closed (see Section 2). Further, we note that the sum in (6) with S is the same as the sum in (4) with $mS^{(-1)}$. Now, let $S = \{1, 4, 18, 30, 36, 45, 60, 90, 180\}$. Then S is lcm-closed and $mS^{(-1)} = \{1, 2, 3, 4, 5, 6, 10, 45, 180\}$, which is the gcd-closed set for (4) given in Section 1. Thus $\det[S] = 0$. This proves Theorem 4. 2

Remark 1 If S is lcm-closed such that $mS^{(-1)}$ is factor-closed, then

$$\det[S] = m^n \prod_{k=1}^{n} \pi(x_k)\phi(x_k)/x_k^2 \tag{7}$$

and thus the LCM matrix $[S]$ is nonsingular.

Theorem 5 *The GCUD matrix $(S)^{**}$ on a gcud-closed set S is always nonsingular.*

Proof It is known [6, Theorem 5.1 with $f = I$] that

$$\det(S)^{**} = \prod_{k=1}^{n} \sum_{d} \phi^*(d), \tag{8}$$

where for each k, d runs through the unitary divisors of x_k such that $d \not\| x_t$ for $x_t < x_k$. Since $\phi^*(d) > 0$ for all d, the GCUD matrix $(S)^{**}$ is nonsingular on gcud-closed sets. 2

Theorem 6 *The SUGCD matrix $(S)^*$ on an sugcd-closed set S is always nonsingular.*

Proof It is known [6, Theorem 5.2 with $f = I$] that

$$\det(S)^* = \prod_{k=1}^{n} \sum_{d} \phi^*(d), \tag{9}$$

where the sum over d is as in (8). Note that the products in (8) and (9) are equal. We thus obtain Theorem 6. 2

Theorem 7 *The LCUM matrix $[S]^{**}$ on a gcud-closed set S is not always nonsingular.*

Proof It is known [6, Theorem 5.3 with $f = I$] that

$$\det[S]^{**} = \prod_{k=1}^{n} x_k^2 \sum_{d} \left(\frac{1}{I} \oplus \mu^*\right)(d), \tag{10}$$

where for each k, d runs through the unitary divisors of x_k such that $d \not\| x_t$ for $x_t < x_k$. It can be verified that

$$\left(\frac{1}{I} \oplus \mu^*\right)(d) = \mu^*(d)\phi^*(d)/d$$

for all d. Thus

$$\det[S]^{**} = \prod_{k=1}^{n} x_k^2 \sum_d \mu^*(d)\phi^*(d)/d. \tag{11}$$

Let $S = \{1, 2, 3, 6, 7, 13, 14, 21, 39, 546\}$. Then S is gcud-closed. Let Σ_n denote the sum in (11) with $x_k = x_n = 546$. Denote briefly $g = \mu^*\phi^*/I$. Then

$$\begin{aligned}\Sigma_n &= g(546) + g(273) + g(182) + g(91) + g(78) + g(42) + g(26) \\ &= \frac{24}{91} - \frac{48}{91} - \frac{36}{91} + \frac{72}{91} - \frac{4}{13} - \frac{2}{7} + \frac{6}{13} \\ &= 0.\end{aligned}$$

This shows that $\det[S]^{**} = 0$. This set was found by calculations using the Mathematica program [9]. 2

Remark 2 If S is ud-closed, then

$$\det[S]^{**} = \prod_{k=1}^{n} x_k \, \mu^*(x_k)\phi^*(x_k) \tag{12}$$

and thus the LCUM matrix $[S]^{**}$ is nonsingular.

Remark 3 Suppose that the elements in S are square-free. Then equations (4) and (11) coincide. In fact, it is easy to see that for each k, d runs through the same divisors of x_k in (4) as in (11). It is likewise easy to see that

$$\pi(d)\phi(d)/d^2 = \mu^*(d)\phi^*(d)/d$$

whenever d is square-free. This shows that the sums in (4) and (11) are equal. Further, it is clear that $(x_i, x_j)^{**} = (x_i, x_j)$ and $[x_i, x_j]^{**} = [x_i, x_j]$. So, equations (4) and (11) coincide. We note that the elements in the set $S = \{1, 2, 3, 6, 7, 13, 14, 21, 39, 546\}$ in the proof of Theorem 7 are square-free. Therefore this set gives a further example for Theorem 2.

Theorem 8 *The GCUD matrix $(S)^{**}$ on an lcum-closed set S is always nonsingular.*

Proof It can be concluded (see [6, Remark 5.7 with $f = I$] and (5)) that

$$\det(S)^{**} = \det(S) = m^{-n} \prod_{k=1}^{n} x_k^2 \sum_d \phi(d), \tag{13}$$

where the sum over d is as in (5). Thus Theorem 8 holds. 2

Theorem 9 *The LCUM matrix $[S]^{**}$ on an lcum-closed set S is not always nonsingular.*

Proof It can be concluded (see [6, Remark 5.7 with $f = I$] and (6)) that

$$\det[S]^{**} = \det[S] = m^n \prod_{k=1}^{n} \sum_{d} \pi(d)\phi(d)/d^2, \tag{14}$$

where the sum over d is as in (5). Let $S = \{1, 14, 26, 39, 42, 78, 91, 182, 273, 546\}$. Then S is lcum-closed and $mS^{(-1)} = \{1, 2, 3, 6, 7, 13, 14, 21, 39, 546\}$, which is the gcud-closed set in the proof of Theorem 7. Since the elements in the sets S and $mS^{(-1)}$ are square-free, the sums over d in (11) and (14) coincide. This shows that $\det[S]^{**} = 0$. 2

Remark 4 If S is lcum-closed such that $mS^{(-1)}$ is ud-closed, then

$$\det[S]^{**} = m^n \prod_{k=1}^{n} \pi(x_k)\phi(x_k)/x_k^2 \tag{15}$$

and thus the LCUM matrix $[S]^{**}$ is nonsingular.

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