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A NEW CONTINUED FRACTION FOR EULER'S CONSTANT

by Aldo Peretti

Abstract:

A new continued fraction is obtained for Euler's constant C, namely:

$$C = \frac{1}{2} - \frac{1}{3} + \left\{ \frac{\left(\frac{2r-1}{r-1}\right)^2 \Big|}{\Big| \frac{-2^r + r}{r(r+1)}} + \frac{2^r + r \,|}{|\, r^2} + r^2 \left\{ \frac{(2^r+m)^2 \,|}{|\, 1} \right\}_{m=1}^{m=2^r-1} \right\}_{r=2}^{r=\infty}$$

It is considered the possibility to prove the irrationality and transcendency of the constant by means of this expansion.

Historical introduction

The early calculation of C was extremely laborious, and can be followed in Glaisher's paper (9).

Euler calculated C in 1781 with 16 digits, but the last was wrong. Mascheroni determined 32 digits, but the last 13 were wrong.

Gauss calculated C with 23 digits; his calculus was checked to be right by Nicolai Gauss, who determined C up to 40 digits.

Oettinger found 43 digits; Shanks 70, of which the last 21 were wrong. Glaisher remarked this error, and evaluated 100 digits. Shanks returned to his former calculations, and gave C to 110 digits, of which 7 were wrong. Adams in 1878 calculated 263 digits, and there was no further progress until 1963.

At the end of the past century, Hermite wrote to Stieltjes (11): " ... as regarding Euler's constant, I can assure you that any human eye, at present, has sounded the mystery of its irrationality. It would be a beautiful and great discovery to prove that C is incommensurable; but from where can come a ray of light about a so deep and hidden question ? ".

According to H. W. Gould (10): " ... it is a familiar story that G. H. Hardy was willing to resign his chair of Mathematics in favour of anyone who could prove C irrational.

Numerous series, integrals, products and other formulas for C have been discovered but none of them have been suitable to prove anything about the arithmetic nature of C. Still the search goes on for new formulas, always with the hope that a valuable new insight will be gained from them. In an unpublished bibliography I have collected together about 150 references on Euler's constant and the many formulas known for this mysterious number ".

In 1926, P. Appell published a proof that C is irrational. Some weeks later he published a retraction after detecting an error. A survey of this proof is given in (1).

The computer's era begun in 1963, when D. Sweeney (13), determined the first 3.566 decimals. His calculations were superseded by W. Beyer and M. Waterman (3, 4), in 1974, and in 1977 by R. Brent (5) who determined 20.700 decimals.

A further computation led R. Brent (6) in 1988, to determine 30.100 decimals and to prove that if the constant is a rational number P / Q, then $Q > 10^{15.000}$.

Work in other direction has been performed by D. Bailey (2) in 1988, who proved that Euler's constant cannot satisfy an algebraic equation with integral coefficients of degree 8 or less, whose norm is $\leq 10^{9}$.

In "Open Questions in Mathematics. A Collection of Unsolved Problems", edited by Dagmar R. Henney, at George Washington University (1981), we find the following quotation:

"In (8) the author claims to have shown that Euler's constant is irrational, but the first paper (7) had a gap. He claims to have fulfilled the gap in the second (8). Is he right?".

The second paper (8) has been reviewed by S. Knapowski, which however does not say if the proof is valid or not.

1. The main result

Let us suppose we wish to prove that numbers such as

[1.1] $$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

[1.2] $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

are irrational ones.

Then we can appeal to Euler's identity (see note (1)):

$$[1.3]\qquad \frac{1}{a_1} - \frac{1}{a_2} + \frac{1}{a_3} - \dots + \frac{(-1)^{n-1}}{a_n} = \cfrac{1}{a_1 + \cfrac{a_1^2}{a_2 - a_1 + \cfrac{a_2^2}{a_3 - a_2 + \dots + \cfrac{a_{n-1}^2}{a_n - a_{n-1}}}}}$$

In the case of the first series [1.1] we have $a_n = \frac{1}{n}$, and we obtain:

$$[1.4]\qquad \log 2 = \cfrac{1}{1 + \cfrac{1^2}{1 + \cfrac{2^2}{1 + \cfrac{3^2}{1 + \dots}}}}$$

In the case of the second series [1.2] we have $a_n = 2\,n + 1$ and we obtain Lord Brouncker's fraction:

$$[1.5]\qquad \frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \dots}}}}$$

The convergence of these developments follows from that of the corresponding series.
Now we use the following theorem:
If in a development such as:

$$[1.6]\qquad K = \cfrac{1}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{\dots}}}}$$

or, in Perron's notation

$$K = \frac{1|}{|a_1} + \dots + \frac{b_n|}{|a_{n+1}} + \dots$$

the "numerators" b_i and the "denominators" a_{i+1} are relatively prime, then the continued fraction represents an irrational number (see note (2)).

From this criterion, and [1.4] and [1.5] follow the irrationality of log 2 and $\pi / 4$.

Now, concerning Euler's constant, we have the following formula, due to S. Selberg (12):

[1.7]
$$\zeta(s) - \frac{1}{s-1} = \sum_{m=2}^{\infty} (-1)^m \frac{2^{s-1} + 4^{s-1} + \dots + 2^{k(s-1)}}{m^s}$$

where $2^k \le m < 2^{k+1}$. (Hence $k = \left[\frac{\log m}{\log 2} \right]$)

The series is convergent for $\sigma > o$.

Putting $s = 1$ in [1.7], and taking into account that

[1.8]
$$\lim_{s \to 1} \zeta(s) - \frac{1}{s-1} = C = \text{Euler's constant}$$

we deduce the following development:

[1.9]
$$C = \sum_{m=2}^{\infty} (-1)^m \frac{k_m}{m}$$

with:

$$k_m = \left[\frac{\log m}{\log 2} \right]$$

We write it down explicitly:

[1.10]

$$C = \frac{1}{2} - \frac{1}{3} + \frac{2}{4} - \frac{2}{5} + \frac{2}{6} - \frac{2}{7} + \frac{3}{8} - \frac{3}{9} + \frac{3}{10} - \frac{3}{11} + \frac{3}{12} - \frac{3}{13} + \frac{3}{14} - \frac{3}{15} \dots$$

The changes in the numerator take place when the denominator is a power of 2, so that it can be written equally well as:

$$[1.11]\quad C = \frac{1}{2} - \frac{1}{3} + 2\left\{\frac{1}{2^2} - \frac{1}{2^2+1} + \frac{1}{2^2+2} - \frac{1}{2^2+3}\right\} +$$

$$3\left\{\frac{1}{2^3} - \frac{1}{2^3+1} + \dots - \frac{1}{2^4-1}\right\} + 4\left\{\frac{1}{2^4} - \frac{1}{2^4+1} + \dots - \frac{1}{2^5+1}\right\} + \dots$$

We consider now the string

$$[1.12]\qquad \dots + r\left\{\frac{1}{2^r} - \frac{1}{2^r+1} + \dots - \frac{1}{2^{r+1}-1}\right\} + \dots$$

and apply to it Euler's formula [1.3]. Due to the fact that a typical term in it is:

$$\frac{\left(\frac{2^r+m}{r}\right)^2 \Big|}{\Big| \frac{2^r+m+1}{r} - \frac{2^r+m}{r}} = \frac{\left(\frac{2^r+m}{r}\right) \Big|}{\Big| \frac{1}{r}}$$

the corresponding span in Euler's development is:

$$[1.13]\qquad \dots + \left\{\frac{\left(\frac{2^r+m}{r}\right)^2 \Big|}{\Big| \frac{1}{r}}\right\}_{m=o}^{m=2^r-1}$$

in obvious notation.

According to a formula in p. 197 of Perron's book (ref. (14)) we have that:

$$[1.14]\quad b_o + \frac{\frac{\alpha_1}{\gamma_1}\Big|}{\Big|\frac{\beta_1}{\delta_1}} + \dots \frac{\frac{\alpha_v}{\gamma_v}\Big|}{\Big|\frac{\beta_v}{\delta_v}} + \dots = b_o + \frac{\delta_1\,\alpha_1 \,|}{|\,\gamma_1\,\beta_1} + \frac{\gamma_1\,\delta_1\,\delta_2\,\alpha_2\,|}{|\,\gamma_2\,\beta_2} + \dots$$

$$+ \frac{\gamma_{v-1}\ \delta_{v-1}\ \delta_v\ \alpha_v \mid}{\mid \gamma_v\ \beta_v} + \dots$$

When applied to [1.13] with $\alpha_v = (2^r + v)$, $\gamma_v = \gamma_{v+1} = \dots = r^2$, $\beta_v = \beta_{v+1} = \dots = 1$, $\delta_v = \delta_{v+1} = \dots = r$, and $b_o = o$ we obtain:

$$[1.15] \quad \left\{ \frac{\left(\frac{2^r + m}{r}\right)^2 \Big|}{\Big| \frac{1}{r}} \right\}_{m=o}^{m=2^r-1} = \frac{2^{2r}\ r \mid}{\mid r^2} + \left\{ \frac{r^4\ (2^r + m)^2 \mid}{\mid r^2} \right\}_{m=1}^{m=2^r-1}$$

On the other hand, in p. 196 of Perron's book we find Satz 2: If some of the two (finite or infinite) continued fractions

$$b_o + \frac{a_1 \mid}{\mid b_1} + \frac{a_2 \mid}{\mid b_2} + \dots + \frac{a_v \mid}{\mid b_v} + \dots$$

and

$$c_o\ b_o + \frac{c_o\ c_1\ a_1 \mid}{\mid c_1\ b_1} + \dots + \frac{c_{v-1}\ c_v\ a_v \mid}{\mid c_v\ b_v} + \dots$$

(where the c_v are arbitrary numbers $\neq o$) converges, then the other converges equally well, and the value of the second is c_o times the value of the first.
We apply this theorem to

$$[1.16] \quad \left\{ \frac{r^4\ (2^r + m)^2 \mid}{\mid r^2} \right\}_{m=1}^{m=2^r-1}$$

with $c_o = c_1 = c_2 = \dots = r^2$, $b_1 = b_2 = \dots = 1$, $b_o = o$.
We deduce that

$$[1.17] \quad \left\{ \frac{r^4\ (2^r + m)^2 \mid}{\mid r^2} \right\}_{m=1}^{m=2^r-1} = r^2 \left\{ \frac{(2^r + m)^2 \mid}{\mid 1} \right\}_{m=1}^{m=2^r-1}$$

Hence, returning to [1.15] we have

$$[1.18] \qquad \left\{ \frac{\left(\frac{2^r+m}{r}\right)^2 \Big|}{\Big| \frac{1}{r}} \right\}_{m=o}^{m=2^r-1} = \frac{2^{2r}\; r \,\big|}{\big|\, r^2} + r^2 \left\{ \frac{(2^r+m)^2 \,\big|}{\big|\, 1} \right\}_{m=1}^{m=2^r-1}$$

Furthermore, the span in [1.13] is preceded in series [1.11] by a transition term.

$$-\frac{(r-1)}{2^r-1}$$

followed by the term $r \,/\, 2^r$. This gives rise in the continued fraction to the transition term (between strings [1.13]):

$$\frac{\frac{(2^r-1)^2}{(r-1)^2} \Big|}{\Big| \frac{2^r}{r} - \frac{2^r-1}{r-1}} = \frac{\left(\frac{2^r-1}{r-1}\right)^2 \Big|}{\Big| -\frac{2^r}{r(r-1)} + \frac{1}{r+1}} = \frac{\left(\frac{2^r-1}{r-1}\right)^2 \Big|}{\Big| \frac{-2^r+r}{r(r+1)}}$$

whose denominator is negative for $r \geq 2$.
To this term follows the string [1.13] so that the whole development has the form.

$$\ldots + \left\{ \frac{\left(\frac{2^r-1}{r-1}\right)^2 \Big|}{\Big| \frac{-2^r+r}{r(r+1)}} + \left\{ \frac{\left(\frac{2^r+m}{r}\right)^2 \Big|}{\Big| \frac{1}{r}} \right\}_{m=o}^{m=2^r-1} \right\} + \ldots =$$

$$= \ldots \left\{ \frac{\left(\frac{2^r-1}{r-1}\right)^2 \Big|}{\Big| \frac{-2^r+r}{r(r+1)}} + \frac{2^{2r}\; r \,\big|}{\big|\, r^2} + \left\{ \frac{r^4\,(2^r+m)^2 \,\big|}{\big|\, r^2} \right\}_{m=1}^{m=2^r-1} \right\} \ldots$$

$$= \cdots \left\{ \frac{\left(\frac{2^r-1}{r-1}\right)^2 \Big|}{\Big|\, \frac{-2^r+r}{r(r+1)}} + \frac{2^{2r}\, r\,\Big|}{\Big|\, r^2} + r^2 \left\{ \frac{(2^r+m)^2\,\Big|}{\Big|\,1} \right\}_{m=1}^{m=2^r-1} \right\} \cdots$$

Hence, we deduce finally that

$$[1.20] \quad C = \frac{1}{2} - \frac{1}{3} + \left\{ \frac{\left(\frac{2^r-1}{r-1}\right)^2 \Big|}{\Big|\, \frac{-2^r+r}{r(r+1)}} + \frac{2^{2r}\, r\,\Big|}{\Big|\, r^2} + r^2 \left\{ \frac{(2^r+m)^2\,\Big|}{\Big|\,1} \right\}_{m=1}^{m=2^r-1} \right\}_{r=2}^{r=\infty}$$

2. The question that arises now, of course, is if development [1.20] can be used to prove either the irrationality or the transcendency of Euler's constant.

As regards the irrationality, we have the theorem already quoted in [1.6] that if the "numerators" and "denominators" are relatively prime, then the fraction represents an irrational number. However, this theorem requires that all the numbers involved be positive.

If it could be extended to the case where most of the denominators are positive, then the irrationality of C could be established immediately.

Concerning the transcendency, we would deduce it very easily if the following theorem (almost self evident) were true: if the continued fraction development of a number N contains infinitely many strings of increasing length of other transcendent number, then N is also transcendental.

It is obvious that the development [1.20] for Euler's constant contains infinite strings of the type:

$$\left\{ \frac{(2^r+m)^2\,\Big|}{\Big|\,1} \right\}_{m=1}^{m=2^r-1}$$

all of them included in the continued fraction development [1.4] for $\log 2$, and $\log 2$ is known to be a transcendental number.

In this way, we have brought both questions, formerly out of our reach, to known fields.

Note 1:

Formula [1.3] appears in ref. (11), but Stieltjes does not indicate the exact place where it appears

in Euler's papers.
It can also be derived from Carr's book, formula 182, putting there $x = -1$.

Note 2:
This is theorem 174, p. 64 of Carr's book (16). The theorem must be due in all likehood, either to Stern or Seidel. It is a surprising fact that the theorem does not appear in Perron's book (ref. (11), (14)), nor in the comprehensive "History of Continued Fractions and Pade Approximants" of C. Brezinski (ref. (15)).
In order to help interested readers in finding the original fountain, the author of this paper has written below an appendix, where are enlisted all the bibliographical references that gives Carr on this subject. Very regrettably, he has no access to such an old bibliography, so he has been unable to push further the question.

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APPENDIX

References about continued fractions quoted in Carr's book:

- A. 18, 33, 55, 69: Archiv der Mathematik, Vol. 18 (1852); Vol. 33 (1859; Vol. 55 (1873); Vol. 69 (1883).

- An. 51: Annali de Scienze Matematiche e Fisiche, compilati da B. Tortolini (1851).

- Gz. C. 99: generalization: C. 99 Comptes rendus hebdomadaires des seances de l'Académie de Sciences. Paris (1884) vol. 99.
  Generalization: CM4: Cambridge Math. Journal. Vol. 4 (1845)
  Theorem: E 30: Educational Times Reprint of Mathematical Questions and Solutions. London. Vol. 30 (1878)
  G 10, 15: Giornale di Matematiche. G. Battaglini. Vol. 10 (1872). Vol. 15 (1877).
  J 6.8, 11_4: (this subscript means 4 papers), and application 18, 53, 80.
  This reference refers to:
    - Journal für die reine und angewandte Mathematik (A. Crelle). Vol. 6 (1830, Vol. 8 (1832, Vol. 11 (1834) (four papers). Applications in: Vol. 18 (1838), Vol. 53 (1857), Vol. 80 (1875).

- L 50, 58, 65: Journal de Mathématiques Pures et Appliqueés (J. Liouville). 1850 (Vol. 15), 1858 (Vo. II, 3), 1865 (Vol. II, 10).

- L.M.S.: London Mathematical Society's Proceeding. Vol. 5 (1874).

- Me 77 (Mem. ap. to i.c. = application to integral calculus): Mémoires de L'Acádemie Impériale des Sciences de Saint Petersburg. (1877) Vol. 23 and 24.

- Mo 66: Monatsbericht der Königlich. Preussischen Akademie der Wissenschaften zu Berlin. (1866) Vol. 11.

- N 42 tr. 49, 56, 66: (tr. means treatise: more than 50 pages)
  N: Nouvelles Annales de Mathématiques
  N 42: (1842) Serie I, N° 1; N 49: (1849) Serie I, N° 8; N 56: (1856) Serie I, N° 15; (1866) Serie II, N° 11.

- Q 4: Quarterly Journal of Mathematics. (1861) Vol. 4.

- Geo Z.12: (geo = geometrical) Zeitschrift für Mathematik und Physik. Vol. 12 (1867)

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I have searched these papers betwen the 2.302 references of Brezinski's book. I have found only the following:

- Cambridge Mathematical Journal. Vol. 4 (1845).
  P. Frost: On certain continued fractions. p. 237-240.

- Journal für die reine und angewandte Mathematik:
  Vol. 6 (1830): A. Möbius: Beitrage zu der Lehre von den Kettenbrüchen, nebst eine Anhänge dioptrischen Inhalts. p. 215-243.
  Vol. 8 (1832): M. Stern: Zur Theorie der periodischen kettenbrüche. p. 1-102.
  Vol. 8 (1832): M. Stern: Uber die Summierung gewisser Kettenbrüche. p. 42-50.
  Vol. 8 (1832): M. Stern: Observationes in fractiones continuas. p. 192-193.
  Vol. 11 (1834): M. Stern: Theorie der Kettenbrüchen und Anwendung
  p. 33-66; p. 142-168; p. 277-36; p. 311-350 (four papers).
  Vol. 18 (1838): M. Stern: Zur Theorie der Kettenbrüchen. p. 69-74.

Vol. 53 (1857): A. Cayley: Note sur l'équation $x^2 - Dy^2 = \pm 4$. p. 369-71

- Nouvelles Annales de Mathématiques
  Serie I, N° 8 (1849): E. Catalan: Théorie des fractions continues. p. 154-202
  Serie I, N° 8 (1849): G. Einsenstein: Sur les fractions continues. p. 341-343

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Carr's book was first published in London under the title "A synopsis of elementary results in pure mathematics", F. Hodgson, 1886.
According to G. H. Hardy, is "A book written with some real scholarship, and with a style and individuality of its own".
It was the basic fountain of learning to the indian mathematician S. Ramanujan.

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