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ON CERTAIN ARITHMETIC FUNCTIONS ASSOCIATED WITH THE UNITARY DIVISORS OF A NUMBER

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0. Preliminaries

The notion of unitary divisor of a number, as well as some arithmetic functions associated with this notion has been introduced by E. Cohen [4,5]. By definition, d is a unitary divisor of n (see also [6]), noted by $d \mid n$, if $d|n$ and $(d, \frac{n}{d}) = 1$. Clearly $1 \mid n$ and $n \mid n$. Let $\sigma_k^*(n)$ be the sum of $k-$th powers of unitary divisors of n, i.e., $\sigma_k^*(n) = \sum_{d|n} d^k$ and $\sigma_1^*(n) = \sigma^*(n), \sigma_0^*(n) = d^*(n)$ - the sum, and the number of unitary divisors of n, respectively. Similarly, let

$$\mu^*(n) = \begin{cases} 1, & \text{if } n = 1 \\ (-1)^r, & \text{if } n = 1 \ (r = \omega(n)) \end{cases}$$

and $\sigma_k^*(n) = n^k . \sum_{d|n} \frac{\mu^*(d)}{d^k}$ be the Möbius and Euler-type arithmetic functions. Define also by $(f \bigoplus g)(n) = \sum_{d|n} f(d).g(\frac{n}{d})$ the unitary convolution of f and g.

Let $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ be the canonical representation of n, where $r = \omega(n)$ denotes the number of distinct prime factors of n. Then we have (cf. [4,5])

$$\sigma_k^*(n) = \prod_{i=1}^{r} (1 + p_i^{k.\alpha_i} \tag{1}$$

$$\sigma^*(n) = \prod_{i=1}^{r} (1 + p_i^{\alpha_i}; d^* = 2^r \tag{2}$$

$$\varphi^*(n) = \prod_{i=1}^{r} (p_i^{k.\alpha_i - 1} \tag{3}$$

and

$$\sigma_k^*(n) = U + E_k, \quad \varphi_k^*(n) = \mu^* + E_k, \tag{4}$$

where we note $E_k(n) = n^k$, $U(n) = 1$ $(n = 1, 2, 3, \ldots$.

The aim of this paper is to prove some relations and inequalities for the above mentioned arithmetical functions as well as connections with the classical arithmetic functions. These relations have similarity with some known results (see [2,6,9,13,14,17,19,22]). For other methods we refer to the paper [21]. For definitions and properties of the so-called "non-unitary divisors", see [12].

1. Inequalities for σ_k^*

1) R. Sivaramakrishnan and C. S. Venkataraman [15] have proved that

$$\sigma_k/d(n) \geq n^{k/2}. \tag{5}$$

On the other hand it is known that

$$\frac{\sigma(n)}{d(n)} \leq \frac{n+1}{2} \tag{6}$$

which is due to E. S. Langford [10,15]. For new proofs and generalizations see [19,20].

We can find an extended analogue of (6) by using the following inequality of G. Polya and G. Szegö [16]: Let $0 < a \leq a_k \leq A$, $0 < b \leq b_k \leq B$ $(k = 1, 2, \ldots, s)$ be two sequences of real numbers. Then

$$\frac{(a_1^2 + a_2^2 + \ldots + a_s^2).(b_1^2 + b_2^2 + \ldots + b_s^2)}{(a_1^2 b_1^2 + a_2^2 b_2^2 + \ldots + a_s^2 b_s^2)} \leq \frac{(AB + ab)^2}{4ABab} \tag{7}$$

To use (7) we first remark that if $d_1, d_2, \ldots, d_s$ are the unitary divisors of $n \geq 2$, then $n|d_1$, $n|d_2$, ... $n|d_s$, are also unitary divisors of n (because of $d|n$, $(d, n/d) = 1$ iff $n/d|n$, $(n/d, n/(n/d)) = 1$) and the equality $\sum_{i=1}^{s} d_i^k = \sum_{i=1}^{s} (n/d_i)^k$ implies that

$$\sum_{i=1}^{s} d_i^{-k} = n^{-k}. \sum_{i=1}^{s} d_i^k. \tag{8}$$

Let $k > 0, l > 0$ and apply (7) for $a_i = d_i^{k/2}, b_i = d_i^{-l/2}$ $(i = 1, 2, \ldots, s)$. Here $a = 1, A = n^{k/2}, B = 1$. Taking into account (8) we get the following inequality:

$$\frac{(\sigma_k^*(n).\sigma_l^*(n))^{1/2}}{\sigma_{\frac{k-l}{2}}^*(n)} \leq n^{\frac{-(k-l)}{4}}.(\frac{n^{\frac{(k+l)}{2}} + 1}{2}), \tag{9}$$

which for $k = l$ yields

$$\frac{(\sigma_k^*(n)}{d_n^*} \le \frac{n^k+1}{2}. \tag{10}$$

This is an extended analogue of Langford's inequality. In order to prove (5) for σ_k^* and d^*, we can apply the well-known inequality $(a_1+a_2+...+a_s).(\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_s}) \ge s^2$ $(a_i > 0)$ for $a_i = d_i^k$. Then, in view of (8) we get

$$\frac{(\sigma_k^*(n)}{d_n^*} \ge n^{k/2}. \tag{11}$$

We can prove a more general relation by using the inequality of Tchebyshev [8]: Let $0 \le a_1^1 ... \le a_s^1, 0 \le a_1^2 ... \le a_s^2, 0 \le a_1^m ... \le a_s^m$ be sequences of real numbers. Then the followinh inequality is true:

$$\frac{\sum_{i=1}^{s} a_i^1}{s}.\frac{\sum_{i=1}^{s} a_i^2}{s}...\frac{\sum_{i=1}^{s} a_i^m}{s} \le \frac{\sum_{i=1}^{s} a_i^1.a_i^2...a_i^m}{s}. \tag{12}$$

Choose $a_i^1 = d_i^{k_1}, ..., a_i^m = d_i^{k_m}$ $(i = 1, ..., s)$, where d_i are the unitary divisors of n, and $k_1, ..., k_m$ are positive real numbers. Then (11) and (12) imply

$$\frac{\sigma_{k_1+...+k_m}^*(n)}{\sigma_{k_i}^*(n)} \ge n^{\frac{1}{2}.\sum_{j\ne i} k_j}. \tag{13}$$

For $m = 2$ we obtain:

$$\frac{\sigma_{k+l}^*(n)}{\sigma_l^*(n)} \ge n^{\frac{k}{2}} \tag{14}$$

which generalizes (11).

The idea of using (12) or more general version of Tchebyshev's inequality is due to J. Sándor [19] and J. Rutkowski [18]. Some results of $\frac{\sigma(n)}{d(n)}$, connected with (6) were also found by P. Laborde [11].

2) An other relation follows from the multiplicative property of σ_k^*, i.e., $\sigma_k^*(m.n) = \sigma_k^*(m).\sigma_k^*(n)$ if $(m,n) = 1$ (see [4,5]). Consider

$$\frac{a^{2n}-1}{a-1} = a^{2n-1} + ... + a + 1 \ge 2n.\sqrt[2n]{a^{\frac{2n(2n-1)}{2}}} = 2n.a^{\frac{2n-1}{2}},$$

i.e. $a^n + 1 \ge \frac{2n.a^{\frac{2n-1}{2}}}{(a^n-1)/(a-1)}$ $(a > 1)$, which applied to $a = p, n = k.\alpha$, leads to (in view of (1))

$$\sigma_k^*(p^\alpha) \ge \frac{2k.\alpha.p^{\frac{2k\alpha-1}{2}}}{\sigma_k(p^{\alpha-1})} \ (p- \text{ prime}) \tag{15}.$$

Let $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ be the canonical representation o fn. Denote $\gamma(n) = p_1.....p_r$ the so-called "core" function of n. Then the multiplicativity of σ_k^*, σ_k and (15) lead to

$$\frac{\sigma_k^*(n)}{d^*(n)} \geq \frac{k^r.\alpha_1.....\alpha_r}{(\gamma(n))^{\frac{1}{2}}.\sigma_k(\frac{n}{\gamma(n)})}.n^k. \tag{16}$$

If n is a squarefree number, i.e., $n = \gamma(n)$ $(\alpha_1 = \alpha_2 = ... = \alpha_r = 1)$ one finds that

$$\frac{\sigma_k^*(n)}{d^*(n)} \geq k^r.n^{k-1/2}. \tag{17}$$

2. Inequalities for φ_k^*

1) The relation $a^n - 1 = (a-1)(a^{n-1} + ... + a + 1) \geq (a-1).n.a^{\frac{n-1}{2}}$ $(a > 1)$ applied to $a = p$ (prime), $n = k.\alpha$, according to (3) implies $\varphi_k^*(p^\alpha) \geq (p-1).k.\alpha.p^{\frac{k\alpha-1}{2}}$. By the multiplicative property of φ_k^* one obtains

$$\varphi_k^*(n) \geq \frac{(p_1-1)....(p_r-1)}{\sqrt{p_1....p_r}}.k^r.\alpha_1.\alpha_2.....\alpha_r.n^{k/2}. \tag{18}$$

By considering the cases $p_1 = 2$ and $p_1 \geq 3$ one arrives to

$$\varphi_k^*(n) \geq \begin{cases} k^r.\alpha_1.\alpha_2.....\alpha_r.n^{k/2}, & \text{for } n \text{ odd} \\ \frac{1}{\sqrt{2}}.k^r.\alpha_1.\alpha_2.....\alpha_r.n^{k/2}, & \text{for } n \text{ even} \end{cases} \tag{19}$$

An interesting consequence of (18) can be obtained by remarking that $\frac{(p_1-1).....(p_r-1)}{p_1.....p_r} = (1-\frac{1}{p_1}).....(1-\frac{1}{p_r})$, and if $p_1 = 2$, then $p_i \geq i+1$ $(i = 1,2,...,r)$; and for $p_1 \geq 3$ one has $p_i \geq i+2$ $(i = 1,2,...,r)$. Then we get $(1-\frac{1}{p_1}).....(1-\frac{1}{p_r}) \geq (1-\frac{1}{2}).....(1-\frac{1}{r+1}) = \frac{1}{r+1}$ $(p_1 = 2)$; and $(1-\frac{1}{p_1}).....(1-\frac{1}{p_r}) \geq (1-\frac{1}{3}).....(1-\frac{1}{r+2}) = \frac{2}{r+2}$ $(p_1 \geq 3)$, respectively.

Since it is well-known that (see [7]): $r = \omega(n) \leq \frac{\log n}{\log 2}$ one easily gets

$$\varphi_k^*(n) \geq \begin{cases} \frac{2.\log 2}{3}.\frac{n^{k/2}}{\log n}.k^r.\alpha_1.\alpha_2.....\alpha_r.(\gamma(n))^{1/2}, & \text{for } n \text{ odd} \\ \frac{\log 2}{2}.\frac{n^{k/2}}{\log n}.k^r.\alpha_1.\alpha_2.....\alpha_r.(\gamma(n))^{1/2}, & \text{for } n \text{ even} \end{cases} \tag{20}$$

This is similar (though more complicated) to a result proved by H. Hatalová and T. Šalát [19].

2) An easy consequence of (4) is

$$\sum_{d|n} \varphi_k^*(d) = n^k. \tag{21}$$

One can prove immediatellt that

$$\text{if } m \mid n, \text{ then } \varphi_k^*(m) \leq \varphi_k^*(n). \tag{22}$$

By using identity (21) and relation (22) we infer that

$$n^k \leq \varphi_k^* d^*(n). \tag{23}$$

A similar result for φ and d due to R. Sivaramakrishnan (see also [23,24]).

3. Inequalities for φ_k^* and σ_k^*

1) One can write

$$\frac{\varphi_k^*(n).\sigma_k^*(n)}{n^{2k}} = \prod_{i=1}^{r} (1 - \frac{1}{p_i^{2k}}), \tag{24}$$

where $n = \prod_{i=1}^{r} p_i^{\alpha_i}$. This identity and $\prod_{i=1}^{r} (1 - \frac{1}{p_i^{2k}}) > \frac{1}{\zeta(2k)}$ (where ζ denotes Riemann's zeta function) permit to deduce

$$\frac{1}{\zeta(2k)} < \frac{\varphi_k^*(n).\sigma_k^*(n)}{n^{2k}} < 1. \tag{25}$$

For $k = 1$ (in the classical case) see A. Makowski [13] and K. Chandrasekharan [3]. On the other hand $p_i \geq i + 1$ yields

$$\frac{\varphi_k^*(n).\sigma_k^*(n)}{n^{2k}} \geq (1 - \frac{1}{2^{2k}}).....(1 - \frac{1}{(r+1)^{2k}}) > (1 - \frac{1}{2^2}).....(1 - \frac{1}{(r+1)^2}) = \frac{r+2}{2.(r+1)} \geq \frac{1}{2}. \tag{26}$$

Notice that (19) and (25) imply

$$\sigma_k^*(n) < \begin{cases} n^{3k/2}, & \text{for } n \text{ odd} \\ \sqrt{2} n^{3k/2}, & \text{for } n \text{ even} \end{cases} \tag{27}$$

For the inequality $\sigma(n) < n^{3/2}$ $(n > 2)$ see [10] and for some refinements V. Annapurna [1].

2) A method of proving arithmetical consists of considering prime powers and using the multiplicative property. Let us prove first that

$$\varphi_k^*(n).(d^*(n))^2 \leq n^{2k} \tag{28}$$

as an analogue of S. Porubski's inequality [17]: $\varphi(n).d^2(n) \leq n^2$, for $n \neq 4$. The functions $\varphi_k^*(n)$, $d^*(n)$, n^{2k} being multiplicative, it is enough to prove (28) for $n = p^\alpha$ (p prime). We have $\varphi_k^*(p^\alpha) = p^{k\alpha} - 1$; $d^*(p^\alpha) = 2$. Then $4.(p^{k\alpha} - 1) \leq p^{2k\alpha}$ iff $(p^{k\alpha} - 2)^2 \leq 0$ with equality for $p = 2, k = \alpha = 1$. Thus we have equality only for $n = 2$.

By the same argument one can prove that

$$\varphi_k^*(n).(d^*(n))^2 > \sigma_k^*(n) \tag{29}$$

which is in connection with a problem of A. Makowski [14]. Indeed, one has $\varphi_k^*(p^\alpha) = p^{k\alpha} - 1$; $d^*(p^\alpha) = 2$; $\sigma_k^*(p^\alpha) = p^{k\alpha} + 1$ and we have to prove that $4.(p^{k\alpha} - 1) > p^{k\alpha} + 1$, i.e. $3.p^{k\alpha} > 5$, obvious.

3) Taking into account that $\sigma_k^* = U \bigoplus E_k, \varphi_k^* = \mu^* \bigoplus E_k$ (see (4)), one has $\sigma_k^*(n) + \varphi_k^*(n) = [(U + \mu^*) \bigoplus E_k](n) \geq 2n^k$ by $1 + \mu^*(m) \geq 0$. Thus

$$\sigma_k^*(n) + \varphi_k^*(n) \geq 2n^k. \tag{30}$$

4) The following simple algebraic inequality will be used:

$$(x_1^m + 1)(x_2^m + 1)...(x_r^m + 1) - (x_1 + 1)^m(x_1 + 1)^m...(x_1 + 1)^m \geq 2^{mr}, \tag{31}$$

where $m, r \geq 1$ are positive integers, and $x_i \geq 2$ $(1 = 1, 2, ..., r)$. This can be proved, e.g., by induction with respect to r (and is left to the interested reader). Let $m = 1$ and $x_i = p_i^{k\alpha_i}$. Then (2),(3),(31) give the inequality:

$$\sigma_k^*(n) \geq \varphi_k^*(n) + d^*(n). \tag{32}$$

Similar to $\sigma(n) \geq \varphi(n) + d(n)$ proved by H. D. Badchi and G. Manoranjan [2]. If we apply (31) for $x_i = p_i^{\alpha_i}, m = k$, we find

$$\sigma_k^*(n) \geq (\varphi^*(n))^k + (d^*(n))^k \tag{33}$$

an unitary analogue of $\sigma_k(n) \geq (\varphi(n))^k + (d(n))^k$ proved by E. Trost [25].

5) Finally, we using a variant of Tchebyshev's inequality and the ideas of proving (13), one can obtain a general result (see [18]). we say that an arithmetical function f is increasing on unitary divisors (or *i.u.d.* function) if the implication

$$\text{if } d \mid n \text{ then } f(d) \leq f(n) \tag{34}$$

holds true for all d, n. Relation (22) shows that φ_k^* is *i.u.d.* ; and similarly, from (1) - (4) immediately follows the same thing for the functions σ_k^*, d^*, E_k.

Let us now suppose that f, g are both $i.u.d.$ functions and let h be a multiplicative and non-negative function. Then the inequality

$$(\sum_{d|n} h(d)).* \sum_{d|n} h(d)f(d)g(d)) \geq (\sum_{d|n} h(d)f(d)).(\sum_{d|n} h(d).g(d)) \tag{35}$$

holds true for every positive integer n.

Selecting $h(n) = U(n)$ in (35) we get

$$d^*(n). \sum_{d|n} f(d)g(d)) \geq (\sum_{d|n} f(d)).(\sum_{d|n} g(d)). \tag{36}$$

Particularly, for $f(d) = d^k, g(d) = d^l$, this inequality dives

$$d^*(n).\sigma^*_{k+l}(n) \geq \sigma^*_k(n).\sigma^*_l(n) \tag{37}$$

which improves (14), if we use relation (11).

We note that if one of f and g is $i.u.d$ and the other one is decreasing on unitary divisors, then (36) is valid with reversed sign of inequality. As an application, one can select $f(n) = n, g(N) = \frac{1}{n}$, and noting that $\sum_{d|n} \frac{1}{d} = \frac{1}{n}. \sum_{d|n} d$, we get exactly relation (11).

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