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REMARKS ON PRIME NUMBERS

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Let $p_1, p_2, \ldots$ be the sequence of the prime numbers (i.e., the sequence 2, 3, 5, ...) and let for every natural number n:

$$\text{б}(n) = \begin{cases} 1, & \text{if } n \text{ is a prime number} \\ 0, & \text{otherwise} \end{cases}$$

where "б" is the second letter (after "a") of the Bulgarian alphabet and the first one which is different from the Roman alphabet letters.

We shall prove the following assertion

THEOREM 1: Let $n \geq 9$ be a natural number. Then

$$p_n > 2.n + \pi(n) - \text{б}(n) - \text{б}(p_n - 2). \qquad (1)$$

Proof: The validity of (1) is seen directly for $n = 9$. Let (1) be valid for some natural number n. We shall prove it for $n + 1$. There exist two cases for this number.

Case 1: $n + 1$ is a prime number. Therefore

$$\text{б}(n + 1) = 1,$$

$$\text{б}(n) = 0,$$

$$\pi(n + 1) = \pi(n) + 1.$$

For $p_{n+1} - 2$ also there exist two cases.

Case 1.1: p_{n+1} is a prime number, i.e., $p_{n+1} = p_n + 2$. Then for

$$A_{n+1} = p_{n+1} - 2.(n + 1) - \pi(n + 1) + \text{б}(n + 1) + \text{б}(p_{n+1} - 2) \qquad (2)$$

it is valid by induction that

$$A_{n+1} = p_n + 2 - 2.(n + 1) - \pi(n) - 1 + 1 + 1$$

$$> p_n - 2.n - \pi(n)$$

$$= p_n - 2.n - \pi(n) + \text{б}(n) + \text{б}(p_n - 2) > 0,$$

because n and $p_n - 2$ are not prime numbers.

Case 1.2: p_{n+1} is not a prime number. Then $p_{n+1} \geq p_n + 4$ and from (2) it follows by induction that:

$$A_{n+1} \geq p_n + 4 - 2.(n + 1) - \pi(n) - 1 + 1$$

$$> p_n - 2.n - \pi(n) + 2$$

$$> p_n - 2.n - \pi(n) + \text{б}(n) + \text{б}(p_n - 2) > 0.$$

Case 2: $n + 1$ is not a prime number. Therefore

$$\text{б}(n + 1) = 0.$$

$$\pi(n + 1) = \pi(n).$$

For $p_{n+1} - 2$ also there exist two cases.

Case 2.1: p_{n+1} is a prime number, i.e., $p_{n+1} = p_n + 2$. Then it is valid by induction that

$$A_{n+1} = p_n + 2 - 2.(n + 1) - \pi(n) + 1$$
$$= p_n - 2.n - \pi(n) + 1$$
$$\geq p_n - 2.n - \pi(n) + \theta(n) + \theta(p_n - 2) > 0,$$

because $p_n - 2$ is not a prime number, i.e. $\theta(p_n - 2) = 0$.

Case 2.2: p_{n+1} is not a prime number. Then $p_{n+1} \geq p_n + 4$ and from (2) it follows by induction that:

$$A_{n+1} \geq p_n + 4 - 2.(n + 1) - \pi(n)$$
$$> p_n - 2.n - \pi(n) + 2$$
$$> p_n - 2.n - \pi(n) + \theta(n) + \theta(p_n - 2) > 0.$$

With which the theorem is proved.

This result is weaker than some of the estimations for p_n from e.g., [1], but there the corresponding estimations are only asymptotic ones. Below we shall discuss another inequation for p_n which is related to the above one.

Obviously for every natural number m there exists a natural number k such that

$$p_m > k + \pi(k).$$

Let the numbers $m \geq 8$ and $k \geq 12$ be fixed. Obviously,

$$p_8 = 19 > 12 + 5 = 12 + \pi(12).$$

Then the following assertion related with the above one is valid.

THEOREM 2: For every natural number $n \geq 1$

$$p_{m+n} > k + 2.n + \pi(k) + \pi(n) - \theta(m + n) - \theta(p_{m+n} - 2). \quad (3)$$

Proof: The validity of (3) is seen for n = 1 as follows:

$$p_{m+1} - k - 2 - \pi(k) - \pi(1) + \theta(m + 1) + \theta(p_{m+1} - 2)$$
$$= p_{m+1} - k - 2 - \pi(k) + \theta(m + 1) + \theta(p_{m+1} - 2)$$
$$> p_{m+1} - p_m + \theta(m + 1) + \theta(p_{m+1} - 2) > 0$$

Let (3) be valid for some natural number n. We shall prove it for n + 1. For this number there exist two cases.

Case 1: m + n + 1 is a prime number. Therefore

$$\theta(m + n + 1) = 1,$$
$$\theta(m + n) = 0,$$
$$\pi(m + n + 1) = \pi(m + n) + 1.$$

For $p_{m+n+1} - 2$ also there exist two cases.

Case 1.1: $p_{m+n+1} - 2$ is a prime number, i.e., $p_{m+n+1} = p_{m+n} + 2$.

Then for

$$A_{n+1} \equiv p_{m+n+1} - k - 2.(n + 1) - \pi(k) - \pi(n + 1) + \delta(m + n + 1) + \delta(p_{m+n+1} - 2) \qquad (4)$$

it is valid (by induction about n) that

$$A_{n+1} = p_{m+n} + 2 - k - 2.(n + 1) - \pi(k) - \pi(n) - 1 + 1 + 1$$
$$> p_{m+n} - k - 2.n - \pi(k) - \pi(n)$$
$$= p_{m+n} - k - 2.n - \pi(k) - \pi(n) + \delta(m + n) + \delta(p_{m+n} - 2) > 0,$$

because m + n and $p_{m+n} - 2$ are not prime numbers.

Case 1.2: p_{m+n+1} is not a prime number. Then $p_{m+n+1} \geq p_{m+n} + 4$

and from (4) it follows (by induction) that:

$$A_{n+1} \geq p_{m+n} + 4 - k - 2.(n + 1) - \pi(k) - \pi(n) - 1 + 1$$
$$> p_{m+n} - k - 2.n - \pi(k) - \pi(n) + 2$$
$$\geq p_{m+n} - k - 2.n - \pi(k) - \pi(n) + \delta(m + n) + \delta(p_{m+n} - 2) > 0.$$

Case 2: n + 1 is not a prime number. Therefore

$$\delta(n + 1) = 0.$$
$$\pi(n + 1) = \pi(n).$$

For $p_{m+n+1} - 2$ also there exist two cases.

Case 2.1: $p_{m+n+1} - 2$ is a prime number, i.e., $p_{m+n+1} = p_{m+n} + 2$.

Then from (4) it is valid that

$$A_{n+1} = p_{m+n} + 2 - k - 2.(n + 1) - \pi(k) - \pi(n) + 1$$
$$= p_{m+n} - k - 2.n - \pi(k) - \pi(n) + 1$$
$$\geq p_{m+n} - k - 2.n - \pi(k) - \pi(n) + \delta(m + n) + \delta(p_{m+n} - 2) > 0,$$

because $p_{m+n} - 2$ is not a prime number, i.e. $\delta(p_{m+n} - 2) = 0$.

Case 2.2: p_{m+n+1} is not a prime number. Then $p_{m+n+1} \geq p_{m+n} + 4$

and from (4) it follows that:

$$A_{n+1} \geq p_{m+n} + 4 - k - 2.(n + 1) - \pi(k) - \pi(n)$$
$$= p_{m+n} - k - 2.n - \pi(k) - \pi(n) + 2$$
$$\geq p_{m+n} - k - 2.n - \pi(k) - \pi(n) + \delta(m + n) + \delta(p_{m+n} - 2) > 0.$$

With which the theorem is proved.

REFERENCES:

[1] Trost E., Primzahlen, Verlag Birkhauser, Basel, 1953.

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