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FOUR EXTREMAL PROBLEMS

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Let everywhere $a_0, a_1, \ldots, a_n$ be an arithmetic progression with a difference d (i.e., $a_i = a_0 + i.d$, for every natural number i), where $d, a_0, a_1, \ldots, a_n \geq 0$.

Four extremal problems and their corollaries are discussed. The introduced here results are extensions and improvements of some results from [1].

THEOREM 1: $\max\limits_{1 \leq k \leq [n/2]} \sum\limits_{i=1}^{k} \sum\limits_{j=k+1}^{n} a_i . a_j$

$$= [\frac{n}{2}].[\frac{n+1}{2}].a_0^2 + [\frac{n}{2}].[\frac{n+1}{2}].(n + 2.[\frac{n}{2}] + 2).a_0.d$$

$$+ \frac{1}{4}.[\frac{n}{2}].[\frac{n+1}{2}].([\frac{n}{2}] + 1).(n + [\frac{n}{2}] + 1).d^2.$$

Proof: Let k be a fixed natural number for which $1 \leq k \leq [\frac{n}{2}]$. Let

$$f(k) = \sum_{i=1}^{k} \sum_{j=k+1}^{n} a_i . a_j .$$

Then

$$f(k) = \sum_{i=1}^{k} (a_i . \sum_{j=k+1}^{n} a_j)$$

$$= \sum_{i=1}^{k} a_i .((n - k).a_0 + k.(n - k).d + \frac{1}{2}.(n - k).(n - k + 1).d)$$

$$= ((n - k).a_0 + \frac{1}{2}.(n - k).(n + k + 1).d).(k.a_0 + \frac{1}{2}.k.(k + 1).d)$$

$$= k.(n - k).a_0^2 + \frac{1}{2}.k.(n - k).(n + 2.k + 2).a_0^2.d +$$

$$\frac{1}{2}.k.(k + 1).(n - k).(n + k + 1).d^2.$$

It can directly be seen that for every k ($1 \leq k \leq [\frac{n}{2}]$) it is valid: $f(k) \leq f([\frac{n}{2}])$. Therefore the maximum is obtained for $k = [\frac{n}{2}]$. From the equality $[\frac{m}{2}] + [\frac{m+1}{2}] = m$, which is valid for every natural number $m \geq 1$, follows the validity of Theorem 1.

COROLLARY 1: $\max\limits_{1 \leq k \leq [n/2]} \sum\limits_{i=1}^{k} \sum\limits_{j=k+1}^{n} i.j$

$$= \frac{1}{4}.[\frac{n}{2}].[\frac{n+1}{2}].([\frac{n}{2}] + 1).(n + [\frac{n}{2}] + 1).$$

The following assertion is proved by the same manner.

THEOREM 2: $\max\limits_{1\le k\le n} \sum\limits_{i=1}^{k} \sum\limits_{j=k+1}^{n} (a_i + a_j)$

$$= 2.[\tfrac{n}{2}].[\tfrac{n+1}{2}].a_0 + \tfrac{1}{2}.[\tfrac{n}{2}].[\tfrac{n+1}{2}].(n + 2.[\tfrac{n}{2}] + 2).d.$$

COROLLARY 2: $\max\limits_{1\le k\le [n/2]} \sum\limits_{i=1}^{k} \sum\limits_{j=k+1}^{n} (i + j)$

$$= \tfrac{1}{2}.[\tfrac{n}{2}].[\tfrac{n+1}{2}].(n + 2.[\tfrac{n}{2}] + 2).$$

THEOREM 3: $\max\limits_{1\le k\le [n/2]} \prod\limits_{i=1}^{k} \prod\limits_{j=k+1}^{n} a_i.a_j$

$$= (\prod_{i=1}^{[n/2]} a_i)^{[n/2]}.(\prod_{i=1}^{[n/2]} a_i)^{n-2.[n/2]}.$$

Proof: Let k be a fixed natural number for which $1 \le k \le [\frac{n}{2}]$. Let

$$f(k) = \prod_{i=1}^{k} \prod_{j=k+1}^{n} a_i.a_j.$$

Then

$$f(k) = \prod_{i=1}^{k} (a_i^{n-k}.\prod_{j=k+1}^{n} a_j) = (\prod_{j=k+1}^{n} a_j)^k.(\prod_{i=1}^{k} a_i)^{n-k}$$

$$= (\prod_{i=1}^{n} a_j)^k.(\prod_{i=1}^{k} a_i)^{n-2.k}.$$

It can directly be seen that for every k ($1 \le k \le [\frac{n}{2}]$) it is valid: $f(k) \le f([\frac{n}{2}])$. Therefore the maximum is obtained for $k = [\frac{n}{2}]$ and the theorem is valid.

COROLLARY 3: $\max\limits_{1\le k\le [n/2]} \prod\limits_{i=1}^{k} \prod\limits_{j=k+1}^{n} i.j = (n!)^{[n/2]}.([\frac{n}{2}]!)^{n-2.[n/2]}.$

Let us define for two natural numbers k and l, for which $k \ge l$:

$$S(k,l) = \sum_{i_1,\ldots,i_l \in \{1,\ldots,k\}} \prod_{j=1}^{l} i_j.$$

LEMMA: For every natural number $k \ge 1$:

$$\prod_{i=1}^{k} a_i = a_0^k + \sum_{i=1}^{k} S(k,i).a_0^{k-i}.d^i.$$

The validity of this assertion follows directly by induction.

THEOREM 4: $\max\limits_{1\le k\le [n/2]} \prod\limits_{i=1}^{k} \prod\limits_{j=k+1}^{n} (a_i + a_j)$

$$= \prod_{i=1}^{[n/2]} ((2.a_0 + (i + [\tfrac{n}{2}]).d)^{[(n+1)/2]}$$

$$+ \prod_{j=1}^{[(n+1)/2]} S([\tfrac{n+1}{2}],j).(2.a_0 + (i + [\tfrac{n}{2}]).d)^{[(n+1)/2]-j}.d^j).$$

Proof: Let k be a fixed natural number for which $1 \le k \le [\frac{n}{2}]$. Let

$$f(k) = \prod_{i=1}^{k} \prod_{j=k+1}^{n} (a_i + a_j).$$

From the Lemma follows that

$$f(k) = \prod_{i=1}^{k} \prod_{j=k+1}^{n} (2.a_0 + (i + j).d)$$

$$= \prod_{i=1}^{k} \prod_{j=1}^{n} ((2.a_0 + (i + k).d) + j.d)$$

$$= \prod_{i=1}^{k} ((2.a_0 + (i + k).d)^{n-k} + \prod_{j=1}^{n-k} S(n-k, j).(2.a_0$$

$$+ (i + k).d)^{n-k-j}.d^{j}).$$

From $k + 1 \le [\frac{n}{2}]$ it follows that

$$\frac{f(k+1)}{f(k)} = \frac{\prod_{i=1}^{k+1} \prod_{j=k+2}^{n} (a_i + a_j)}{\prod_{i=1}^{k} \prod_{j=k+1}^{n} (a_i + a_j)} = \frac{\prod_{j=k+2}^{n} (a_{k+1} + a_j)}{\prod_{i=1}^{k} (a_i + a_{k+1})} > 1.$$

From where, the validity of the Theorem is follows.

COROLLARY 4: $$\max_{1 \le k \le [n/2]} \prod_{i=1}^{k} \prod_{j=k+1}^{n} (i + j).$$

$$= \prod_{i=1}^{[n/2]} ((i + [\frac{n}{2}])^{[(n+1)/2]}$$

$$+ \prod_{j=1}^{[(n+1)/2]} S([\frac{n+1}{2}], j).(i + [\frac{n}{2}])^{[(n+1)/2]-j}).$$

We must note that a formula for $\max_{1 \le k \le [n/2]} \prod_{i=1}^{k} \prod_{j=k+1}^{n} (i + j)$ can be obtained directly and it has a simpler form:

$$\max_{1 \le k \le [n/2]} \prod_{i=1}^{k} \prod_{j=k+1}^{n} (i + j) = \prod_{i=1}^{[n/2]} \frac{(n + i)!}{([n/2] + i)!}.$$

From where it follows that for every i $(1 \le i \le [\frac{n}{2}])$:

$$(i + [\frac{n}{2}])^{[(n+1)/2]} + \prod_{j=1}^{[(n+1)/2]} S([\frac{n+1}{2}], j).(i + [\frac{n}{2}])^{[(n+1)/2]-j}$$

$$= \prod_{i=1}^{[n/2]} \frac{(n + i)!}{([n/2] + i)!}.$$

REFERENCE:

[1] Atanassov K., Seven extremal problems, Proc. of International School on Automation and Scientific Research, Varna, 1982, 137-139 (in Bulgarian).

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