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TWO EXTREMAL PROBLEMS

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Let $a_0, a_1, \dots, a_n$ be an arithmetic progression with a difference d (i.e., $a_s = a_0 + s.d$), let $(k_1, k_2, \dots, k_n)$ be an arbitrary permutation of (1, 2, ..., n), let [x] be the integer part, and let |x| be the modulus of the real number x.

Two extremal problems and their corollaries are discussed. The introduced here results are extensions of some results from [1].

Without loss of generality we can assume that d > 0.

THEOREM 1: For every natural number n:

$$\max \sum_{i=1}^{n} |a_i - a_{k_i}| = [\frac{n^2}{2}].d, \tag{1}$$

where the maximum is determined for all permutations $(k_1, k_2, \dots, k_n)$ of (1, 2, ..., n).

Proof: First we shall discuss one particular case of the above sum. Let

$$A_n = \sum_{i=1}^{n} |a_i - a_{l_i}|, \tag{2}$$

where $l_i = n + 1 - i$. Then

$$A_n = \sum_{i=1}^{n} |(a_0 + i.d) - (a_0 + (n + 1 - i).d)|$$

$$= \sum_{i=1}^{n} |(2.i - n - 1).d)|$$

$$= d.\sum_{i=1}^{n} |2.i - n - 1|,$$

and obviouly,

$$|2.i - n - 1| = |n + 1 - 2.i|.$$

There are two cases for n: n is an odd and n is an even number.

Let n = 2.m for a certain natural number m. Then

$$A_{2m} = d.\sum_{i=1}^{2.m} |2.i - 2.m - 1| = 2.d.m^2 = d.\frac{n^2}{2}.$$

Let n = 2.m + 1 for a certain natural number m. Then

$$A_{2m+1} = d.\sum_{i=1}^{2.m+1} |2.i - 2.m - 2| = 2.d.m.(m+1) = d.\frac{n^2 - 1}{2}.$$

Obviously,

$$[\frac{n^2}{2}] = \begin{cases} \frac{n^2}{2}, & \text{if n ia an even number} \\ \frac{n^2 - 1}{2}, & \text{if n ia an odd number} \end{cases}$$

Therefore from (2) follows that

$$A_n = [\frac{n^2}{2}].d. \tag{3}$$

Let

$$B_n = \sum_{i=1}^{n} |a_i - a_{k_i}| \tag{4}$$

for a certain permutation $(k_1, k_2, \ldots, k_n)$ of $(1, 2, \ldots, n)$. We shall prove, that

$$B_n \leq A_n. \tag{5}$$

Let $n = 1$. The validity of (5) is obvious. Let us assume that (5) is valid for a certain natural number n. We shall note, that if a certain member of the permutation $(k_1, k_2, \ldots, k_{n+1})$ is fixed then for B_{n+1} is valid the inequality (see (4)):

$$B_{n+1} = B_n + (a_{n+1} - a_{k_{n+1}})$$

$$= B_n + (a_0 + (n+1).d - (a_0 + k_{n+1}.d))$$

$$\leq B_n + d.n,$$

i.e.

$$B_{n+1} \leq B_n + d.n. \tag{6}$$

Let $n = 2.m$. From (6), (5) and (3) follows that

$$B_{n+1} \leq B_n + n \leq A_n + n = \frac{n^2}{2} + n = \frac{(n+1)^2}{2} = A_{n+1}.$$

Analogically, let $n = 2.m + 1$. Then

$$B_{n+1} \leq B_n + n \leq A_n + n = \frac{n^2 - 1}{2} + n < \frac{(n+1)^2}{2} = A_{n+1}.$$

Therefore (5) is valid and hence the validity of (1) is proved.

<u>CORALLARY 1 [1]</u>: For every natural number n:

$$\max \sum_{i=1}^{n} |i - k_i| = [\frac{n^2}{2}].$$

The second problem is similar to the first one.

Let

$$n!! = \begin{cases} 2.4. \ldots .n, & \text{if n is an even number} \\ 1.3. \ldots .n, & \text{if n is an odd number} \end{cases}.$$

<u>THEOREM 2:</u> For every natural number n:

$$\max \prod_{i=1}^{n} |a_i - a_{k_i}| = \frac{\frac{1 + (-1)^n}{2} + 1}{2} . ((n - 1)!!)^2 . d^n .$$

where the maximum is determined for all permutations $(k_1, k_2, \ldots, k_n)$ of $(1, 2, \ldots, n)$.

Proof: Let $n = 2.m$, where m is a certain natural number. Let

$$C_n = \prod_{i=1}^{n} |a_i - a_{l_i}|, \qquad (7)$$

where $l_i = n + 1 - i$. Then

$$C_{2.m} = \prod_{i=1}^{2.m} |(a_0 + i.d) - (a_0 + (2.m + 1 - i).d)|$$

$$= \prod_{i=1}^{2.m} |(2.i - 2.m - 1).d|$$

$$= d^{2.m} . \sum_{i=1}^{2.m} |2.i - 2.m - 1|,$$

$$= ((n - 1)!!)^2 . d^n .$$

Let

$$D_n = \prod_{i=1}^{n} |a_i - a_{k_i}| \qquad (8)$$

for a certain permutation $(k_1, k_2, \ldots, k_n)$ of $(1, 2, \ldots, n)$. We shall prove, that

$$D_n \le C_n . \qquad (9)$$

Let $m = 1$. The validity of (9) is obvious. Let us assume that (9) is valid for a certain natural number m. From

$$|(2.m + 2) - 1|.|(2.m+ 1) - 2| = 4.m^2 - 1 < 4.m^2$$
$$= |(2.m + 2) - 2|.|(2.m + 1) - 1|$$

follows that

$$D_{2m+2} < D_{2.m} . (2.m)^2 . d^2 \le C_{2.m} . (2.m)^2$$

$$\le ((2.m - 1)!!)^2 . d^{2.m} . (2.m)^2 . d^2$$

$$< ((2.m + 1)!!)^2 . d^{2.m+2} = C_{2m+2},$$

i.e. (9) is valid.

Let $n = 2.m + 1$, where m is a certain natural number. Let

$$E_n = \prod_{i=1}^{m-1} |a_i - a_{l_i}|.|(m - 1) - (m + 1)|.|m - (m - 1)|.$$

$$|(m + 1) - m|. \prod_{i=m+2}^{2.m+1} |a_i - a_{l_i}|$$

where $l_i = n + 1 - i$. Then

$$E_{2.m+1} = 2.d^3 . \prod_{i=1}^{m-1} |(a_0 + i.d) - (a_0 + (2.m + 1 - i).d)|.$$

$$\prod_{i=m+1}^{2.m+1} |(a_0 + i.d) - (a_0 + (2.m + 1 - i).d)|$$

$$= 2.d^3.(\prod_{i=1}^{m-1} |(2.i - 2.m).d|)^2$$

$$= 2.d^{2.m+1}.(\sum_{i=1}^{m-1} |2.i - 2.m - 1|)^2$$

$$= 2.4^2.6^2. \ldots (2.m)^2.d^{2.m+1}$$

$$= \frac{((n-1)!!)^2}{2}.d^n.$$

Let

$$F_n = \prod_{i=1}^{n} |a_i - a_{k_i}| \qquad (10)$$

for a certain permutation $(k_1, k_2, \ldots, k_n)$ of $(1, 2, \ldots, n)$. We shall prove, that

$$F_n \le E_n. \qquad (11)$$

Let $m = 0$. The validity of (11) is obvious. Let us assume that (11) is validfor a certain natural number m. From

$$F_{2m+3} \le F_{2.m+1}.(2.m + 2)^2.d^2$$

and form (11) it follows that

$$F_{2.m+3} \le E_{2.m+1}.(2.m + 2).d^2$$

$$= \frac{((2.m)!!)^2}{2}.d^{2.m+3}.(2.m + 2)!!)^2 = E_{2.m+3},$$

i.e. (11) is valid. With this the validity of the theorem is proved, because

$$\frac{\frac{1 + (-1)^n}{2} + 1}{2} = \begin{cases} 1, & \text{if n is an even number} \\ \frac{1}{2}, & \text{if n is an even number} \end{cases}.$$

<u>CORALLARY 2 [1]</u>: For every natural number n:

$$\max \prod_{i=1}^{n} |i - k_i| = \frac{\frac{1 + (-1)^n}{2} + 1}{2}.((n-1)!!)^2.$$

REFERENCE:

[1] Atanassov K., Three extremal problems, Scientific Session of VNVU "V. Levski", V. Tarnovo, 1984, 285-289.

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